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Constructing the physical parameters of a damped vibrating system from eigendata

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Abstract

In this paper we consider the inverse problem for a discrete damped mass–spring system where the mass, damping, and stiffness matrices are all symmetric tridiagonal. We first show that the model can be constructed from two real eigenvalues and three real eigenvectors or two complex conjugate eigenpairs and a real eigenvector. Then, we study the general under-determined and over-determined problems. In particular, we provide the sufficient and necessary conditions on the given two real or complex conjugate eigenpairs so that the under-determined problem has a physical solution. However, for large model order, the construction from these data may be sensitive to perturbations. To reduce the sensitivity, we propose the minimum norm solution over the under-determined noisy data and the least squares solution to the over-determined measured data. We also discuss the physical realizability of the required model by the positivity-constrained regularization method for the ill-posed under-determined problem and the least squares optimization problems with positivity-constraints for the ill-posed over-determined problem. Finally, we give simple numerical examples to illustrate the effectiveness of our methods.

Keywords: Inverse problem; Vibration system; Ill-posedness; Regularization

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1. Introduction

A free vibrating system with n degrees of freedom is governed by the equation of the form:

$$P(\lambda)\mathbf{x} = \mathbf{0} \tag{1}$$

with the quadratic pencil $P(\lambda)$ defined by

$$P(\lambda) \equiv \lambda^2 M + \lambda C + K, \tag{2}$$

where the matrices M, C, and K are all real symmetric tridiagonal:

$$M = \begin{bmatrix} 2m_1 + 2m_2 & m_2 & & & \\ m_2 & 2m_2 + 2m_3 & m_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & m_n & 2m_n \end{bmatrix}, \tag{3}$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 & & & \\ -c_2 & c_2 + c_3 & -c_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -c_n & c_n \end{bmatrix}, \tag{4}$$

and

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -k_n & k_n \end{bmatrix}. \tag{5}$$

This is a damped mass–spring model, where the matrices M, C, and K are called the mass, damping, and stiffness matrices, respectively. The real numbers $(m_j)_1^n$, $(c_j)_1^n$, and $(k_j)_1^n$ denote the physical parameters with the additional requirements that these parameters should be positive in real-life structures. For general vibrating systems, see for instance [14,15]. It is very known that (1) is also a special kind of quadratic eigenvalue problem (QEP). The scalar λ and the corresponding nonzero vector $\mathbf{u}$ are called, respectively, eigenvalue and eigenvector of the QEP (1). For the applications, mathematical properties, and various numerical solution techniques of general QEPs, see [30] and the references therein.

In this paper, we consider the inverse problem for the vibrating system (1). That is, we reconstruct the quadratic pencil (2) from given eigenvalue/eigenvector data.

The inverse problems can be stated as follows:

Problem A. Construct the parameters $(m_j, c_j, k_j)_1^n$ from $w = \sum_1^n m_j$ and two real eigenvalues λ, ϕ and three real eigenvectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$.

Problem B. Construct the parameters $(m_j, c_j, k_j)_1^n$ from $w = \sum_1^n m_j$ and one real eigenvector $\mathbf{x}$ and two complex conjugate eigenpairs $(\alpha + \beta\mathrm{i}, \mathbf{x}_R + \mathbf{x}_R\mathrm{i})$ and $(\alpha - \beta\mathrm{i}, \mathbf{x}_R - \mathbf{x}_I\mathrm{i})$, where $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$, $\mathbf{x}_R, \mathbf{x}_I \in \mathbb{R}^n$, and $\mathrm{i} = \sqrt{-1}$.

Problem C. Construct the parameters $(m_j, c_j, k_j)_1^n$ from $w = \sum_1^n m_j$ and k eigenpairs $\{(\lambda_p, \mathbf{x}^{(p)})\}_{p=1}^k$ where $\lambda_{2j-1} := \alpha_j + \beta_j\mathrm{i}, \mathbf{x}^{(2j-1)} := \mathbf{x}_R^{(j)} + \mathbf{x}_I^{(j)} i$ and $\lambda_{2j} := \alpha_j - \beta_j\mathrm{i}, \mathbf{x}^{(2j)} := \mathbf{x}_R^{(j)} - \mathbf{x}_I^{(j)} i$ are complex-valued for $j = 1, 2 \ldots, \ell$ and $(\lambda_j, \mathbf{x}^{(j)})$ are real-valued for $j = 2\ell + 1, \ldots, k$.

Inverse eigenvalue problems arise in a remarkable variety of applications. This includes control design, inverse Sturm–Liouville problems, applied mechanics, applied physics, and signal and data processing, etc., see for instance [7,8,9,32] and the references therein.

Recently, inverse eigenvalue problems for vibrating systems have been obtained considerable discussions. [14] is a comprehensive reference for these problems. There is a large literature on the construction of the quadratic pencil (2) from eigenvalue/eigenvector data in many special cases. The problem of reconstructing a Jacobi matrix J from its spectral data (it is a special case of (2) with $M = \mathrm{diag}(m_1, \ldots, m_n)$, $C = 0$, and $J \equiv M^{-1/2}KM^{-1/2}$) has received much attention in the literature, e.g., [3,4,14,17,20]. Ram [23] studied the problem of reconstructing the undamped mass–spring system (i.e. the pencil (2) with $M = \mathrm{diag}(m_1, \ldots, m_n)$ and $C = 0$) from two spectra and Ram [24] considered the same problem from a single eigenvalue, two eigenvectors, and the total mass. Ram and Gladwell [26] generalized the problem to finite element model with tridiagonal mass matrix (i.e., the quadratic pencil (2) with $C = 0$). Ram and Elhay [25] discussed the reconstruction of the symmetric tridiagonal quadratic pencil $P(\lambda) = \lambda^2 I_n + \lambda\widetilde{C} + \widetilde{K}$ (a special case of (2) with $M = \mathrm{diag}(m_1, \ldots, m_n)$, $\widetilde{C} = M^{-1/2}CM^{-1/2}$ and $\widetilde{K} = M^{-1/2}KM^{-1/2}$) from two spectra.

In this paper, we will first solve Problems A and B by construction. Then, we extend to the general problem (i.e., Problem C) over any given eigendata. In particular, we present the sufficient and necessary conditions on the two real or complex conjugate eigenpairs so that the under-determined problem has a physically realizable solution. However, the constructed models may be sensitive to noise. The eigendata is often experimentally measured from the physical structures, which are corrupted by noise [12]. To reduce the sensitivity of the problems, by using these measured eigenpairs, we will consider the over-determined problem in the least squares sense and the under-determined problem in the minimum norm sense. Furthermore, based on the under-determined noisy eigendata, we propose the regularization method for finding a stable and physically feasible solution. Also, based on the over-determined noisy data, the positivity of the parameters $(m_j, c_j, k_j)_1^n$ is obtained by the solution of a set of positivity-constrained least-squares optimization problems.

This paper is organized as follows. We directly construct the solutions to Problems A and B in Sections 2 and 3, respectively. Then, in Section 4 we consider Problem C and provide the solvability conditions on the given two real or complex conjugate eigenpairs so that the under-determined problem has a physical solution. In Section 5 we consider the ill-posed under-determined and over-determined problems over the given experimentally measured data. Different numerical methods are presented for different cases. The physical realizability of the required model is also discussed. In Section 6 we present some numerical results to illustrate the effectiveness of proposed methods. Finally, we give some concluding remarks in Section 7.

2. Problem A

In this section, we will discuss the solvability conditions for Problem A and propose a constructive procedure for solving Problem A. Given three real eigenpairs: $(\lambda, \mathbf{x})$, $(\phi, \mathbf{y})$, and $(\psi, \mathbf{z})$. By (1), we have

$$\begin{cases} (\lambda^2 M + \lambda C + K)\mathbf{x} = \mathbf{0}, \\ (\phi^2 M + \phi C + K)\mathbf{y} = \mathbf{0}, \\ (\psi^2 M + \psi C + K)\mathbf{z} = \mathbf{0}. \end{cases} \tag{6}$$

Since the associated matrices M, C, and K have the structures as in (3)–(5), respectively, we can rewrite (6) as the following $3n$ pairs of equations

$$\begin{cases} \lambda^2 a_n^x m_n + \lambda d_n^x c_n + d_n^x k_n = 0, \\ \phi^2 a_n^y m_n + \phi d_n^y c_n + d_n^y k_n = 0, \\ \psi^2 a_n^z m_n + \psi d_n^z c_n + d_n^z k_n = 0, \end{cases} \tag{7}$$

and

$$\begin{cases} \lambda^2 a_j^x m_j + \lambda d_j^x c_j + d_j^x k_j + \lambda^2 b_j^x m_{j+1} - \lambda d_{j+1}^x c_{j+1} - d_{j+1}^x k_{j+1} = 0, \\ \phi^2 a_j^y m_j + \phi d_j^y c_j + d_j^y k_j + \phi^2 b_j^y m_{j+1} - \phi d_{j+1}^y c_{j+1} - d_{j+1}^y k_{j+1} = 0, \\ \psi^2 a_j^z m_j + \psi d_j^z c_j + d_j^z k_j + \psi^2 b_j^z m_{j+1} - \psi d_{j+1}^z c_{j+1} - d_{j+1}^z k_{j+1} = 0 \end{cases} \tag{8}$$

for $j = 1, \ldots, n-1$, where

$$\begin{cases} (a_j^x)_1^n = x_{j-1} + 2x_j, & (b_j^x)_1^{n-1} = 2x_j + x_{j+1}, & (d_j^x)_1^n = x_j - x_{j-1}, \\ (a_j^y)_1^n = y_{j-1} + 2y_j, & (b_j^y)_1^{n-1} = 2y_j + y_{j+1}, & (d_j^y)_1^n = y_j - y_{j-1}, \\ (a_j^z)_1^n = z_{j-1} + 2z_j, & (b_j^z)_1^{n-1} = 2z_j + z_{j+1}, & (d_j^z)_1^n = z_j - z_{j-1} \end{cases} \tag{9}$$

with $x_0 = y_0 = z_0 = 0$.

Suppose that the total mass $w = \sum_1^n m_j (m_n \neq 0)$, the two real numbers $\{\lambda, \phi\}$, and the two real eigenvectors $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ are given. Our goal is to seek the solution $(m_j, c_j, k_j)_1^n$ in terms of these eigendata. Toward this end, we rewrite Eqs. (7)–(8) in the matrix form. For $j = 1, 2, \ldots, n$, define

$$\tilde{m}_j = m_j/m_n, \quad \tilde{c}_j = c_j/m_n, \quad \tilde{k}_j = k_j/m_n. \tag{10}$$

By definition, $\tilde{m}_n = 1$. In particular, we can find $\tilde{c}_n$ and $\tilde{k}_n$ by (7), i.e.,

$$\begin{bmatrix} \lambda d_n^x & d_n^x \\ \phi d_n^y & d_n^y \\ \psi d_n^z & d_n^z \end{bmatrix} \begin{pmatrix} \tilde{c}_n \\ \tilde{k}_n \end{pmatrix} = - \begin{pmatrix} \lambda^2 a_n^x \\ \phi^2 a_n^y \\ \psi^2 a_n^z \end{pmatrix}. \tag{11}$$

Since we are interested in the nontrivial solution of (11), the real number ψ is determined by

$$\det \begin{bmatrix} \lambda^2 a_n^x & \lambda d_n^x & d_n^x \\ \phi^2 a_n^y & \phi d_n^y & d_n^y \\ \psi^2 a_n^z & \psi d_n^z & d_n^z \end{bmatrix} = 0 \tag{12}$$

such that

$$\operatorname{rank}\left(\begin{bmatrix} \lambda d_n^x & d_n^x \\ \phi d_n^y & d_n^y \\ \psi d_n^z & d_n^z \end{bmatrix} \right) = \operatorname{rank}\left(\begin{bmatrix} \lambda^2 a_n^x & \lambda d_n^x & d_n^x \\ \phi^2 a_n^y & \phi d_n^y & d_n^y \\ \psi^2 a_n^z & \psi d_n^z & d_n^z \end{bmatrix} \right).$$

Next, dividing expression (8) by m_n gives rise to

$$\begin{bmatrix} \lambda^2 a_j^x & \lambda d_j^x & d_j^x \\ \phi^2 a_j^y & \phi d_j^y & d_j^y \\ \psi^2 a_j^z & \psi d_j^z & d_j^z \end{bmatrix} \begin{pmatrix} \tilde{m}_j \\ \tilde{c}_j \\ \tilde{k}_j \end{pmatrix} = \begin{bmatrix} -\lambda^2 b_j^x & \lambda d_{j+1}^x & d_{j+1}^x \\ -\phi^2 b_j^y & \phi d_{j+1}^y & d_{j+1}^y \\ -\psi^2 b_j^z & \psi d_{j+1}^z & d_{j+1}^z \end{bmatrix} \begin{pmatrix} \tilde{m}_{j+1} \\ \tilde{c}_{j+1} \\ \tilde{k}_{j+1} \end{pmatrix} \tag{13}$$

for $j = n-1, \ldots, 1$.

To show our main theorem, define

$$\mathbf{g}^{(n)} = - \begin{pmatrix} \lambda^2 a_n^x \\ \phi^2 a_n^y \\ \psi^2 a_n^z \end{pmatrix}, \quad \widetilde{\mathbf{w}}^{(n)} = \begin{pmatrix} \tilde{c}_n \\ \tilde{k}_n \end{pmatrix}, \quad \widetilde{\mathbf{w}}^{(j)} = \begin{pmatrix} \tilde{m}_j \\ \tilde{c}_j \\ \tilde{k}_j \end{pmatrix} \quad \text{for } 1 \leqslant j \leqslant n-1, \tag{14}$$

$$A_{nn} = \begin{bmatrix} \lambda d_n^x & d_n^x \\ \phi d_n^y & d_n^y \\ \psi d_n^z & d_n^z \end{bmatrix}, \quad A_{jj} = \begin{bmatrix} \lambda^2 a_j^x & \lambda d_j^x & d_j^x \\ \phi^2 a_j^y & \phi d_j^y & d_j^y \\ \psi^2 a_j^z & \psi d_j^z & d_j^z \end{bmatrix} \quad \text{for } 1 \leqslant j \leqslant n-1,$$

and

$$B_{jj} = \begin{bmatrix} -\lambda^2 b_j^x & \lambda d_{j+1}^x & d_{j+1}^x \\ -\phi^2 b_j^y & \phi d_{j+1}^y & d_{j+1}^y \\ -\psi^2 b_j^z & \psi d_{j+1}^z & d_{j+1}^z \end{bmatrix} \quad \text{for } 1 \leqslant j \leqslant n-1.$$

Based on the above analysis, we provided a sufficient and necessary condition for the solvability of Problem A as follows.

Theorem 2.1. *Problem A has a nontrivial solution if and only if the following conditions are satisfied*:

(1) *The real number* ψ *is determined by* (12) *such that* $\mathrm{rank}(A_{nn}) = \mathrm{rank}([A_{nn}, \mathbf{g}^{(n)}])$;
(2) $\mathrm{rank}(A_{jj}) = \mathrm{rank}([A_{jj}, B_{jj}\tilde{\mathbf{w}}^{(j+1)}])$ *for* $j = n-1, \ldots, 1$.

Proof. Problem A has a nontrivial solution if and only if Eq. (11) has a nontrivial solution and (13) has a nontrivial solution successively for $j = n-1, \ldots, 1$, i.e., if and only if conditions (1) and (2) of Theorem 2.1 are satisfied. $\square$

Corollary 2.2. *Problem A has a unique nontrivial solution if and only if the following conditions are satisfied*:

(1) *The real number* ψ *is determined by* (12) *such that* $\mathrm{rank}(A_{nn}) = \mathrm{rank}([A_{nn}, \mathbf{g}^{(n)}]) = 2$;
(2) $\det(A_{jj}) \neq 0$, $(j = n-1, n-2, \ldots, 1)$.

Remark 2.3. Under the conditions of Theorem 2.1 or Corollary 2.2, one may first get $\tilde{c}_n, \tilde{k}_n$ by (11) and then apply Eqs. (13) successively for $j = n-1, n-2, \ldots, 1$ to determine $(\tilde{m}_j, \tilde{c}_j, \tilde{k}_j)$ for $j = n-1, n-2, \ldots, 1$ in turn. Finally, since the total mass $w = \sum_1^n m_j$ is known and one can calculate $\tilde{w} = \sum_1^n \tilde{m}_j$. By (10), the parameters $(m_j, c_j, k_j)_1^n$ are given by

$$m_j = \tilde{m}_j w/\tilde{w}, \quad c_j = \tilde{c}_j w/\tilde{w}, \quad k_j = \tilde{k}_j w/\tilde{w}. \tag{15}$$

In practice, by our procedure, one may find some of the parameters $(m_j, c_j, k_j)_1^n$ are not positive, which is not physical realizable. However, it seems not easy to find a necessary and sufficient condition on the eigendata so that the constructed solution is physical feasible. This needs further research.

For the purpose of demonstration, we present the following example.

2.1. Example 1

Let $n = 3$ and we randomly generate w, $\{\lambda, \phi\}$ and $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ as follows:

$$\begin{cases} w = 1.0403, \quad \lambda = -4.6693, \quad \phi = -2.1036, \\ \mathbf{x} = (-0.2063, -0.1469, 0.2142)^{\mathrm{T}}, \\ \mathbf{y} = (0.4754, 0.2639, -0.4085)^{\mathrm{T}}, \\ \mathbf{z} = (0.5900, -0.5648, 0.5853)^{\mathrm{T}}. \end{cases}$$

Then we have that

$$\begin{cases}(a_j^x)_1^3 = (-0.4125, -0.5000, 0.2814)^{\mathrm{T}},\\ (b_j^x)_1^2 = (-0.5594, -0.0796)^{\mathrm{T}}, \quad (d_j^x)_1^3 = (-0.2063, 0.0594, 0.3610)^{\mathrm{T}},\\ (a_j^y)_1^3 = (0.9508, 1.0031, -0.5531)^{\mathrm{T}},\\ (b_j^y)_1^2 = (1.2146, 0.1193)^{\mathrm{T}}, \quad (d_j^y)_1^3 = (0.4754, -0.2115, -0.6724)^{\mathrm{T}},\\ (a_j^z)_1^3 = (1.1799, -0.5397, 0.6058)^{\mathrm{T}},\\ (b_j^z)_1^2 = (0.6151, -0.5443)^{\mathrm{T}}, \quad (d_j^z)_1^3 = (0.5900, -1.1548, 1.1501)^{\mathrm{T}}.\end{cases}$$

First, ψ is determined by Eq. (12), i.e.,

$$\det\begin{bmatrix} 6.1362 & -1.6859 & 0.3610\\ -2.4474 & 1.4144 & -0.6724\\ 0.6058\cdot\psi^2 & 1.1501\cdot\psi & 1.1501\end{bmatrix} = 0. \tag{16}$$

By (16), we get two feasible solutions: $\psi = -1.6950$ or $\psi = -8.1870$. For $\psi = -1.6950$, by (11), we have

$$\begin{bmatrix} -1.6859 & 0.3610\\ 1.4144 & -0.6724\end{bmatrix}\begin{pmatrix}\tilde{c}_3\\ \tilde{k}_3\end{pmatrix} = \begin{pmatrix}-6.1362\\ 2.4474\end{pmatrix}. \tag{17}$$

We find from (17) that $\tilde{c}_3 = 5.2053$ and $\tilde{k}_3 = 7.3098$. Setting $j = 2$ in Eq. (13) gives

$$\begin{bmatrix} -10.9019 & -0.2772 & 0.0594\\ 4.4390 & 0.4449 & -0.2115\\ -1.5505 & 1.9574 & -1.1548\end{bmatrix}\begin{pmatrix}\tilde{m}_2\\ \tilde{c}_2\\ \tilde{k}_2\end{pmatrix} = \begin{pmatrix}-4.4006\\ 1.9196\\ -0.1768\end{pmatrix},$$

which yields that $\tilde{m}_2 = 0.3163$, $\tilde{c}_2 = 5.3028$, $\tilde{k}_2 = 8.7168$. For $j = 1$, Eq. (13) leads to

$$\begin{bmatrix} -8.9939 & 0.9631 & -0.2063\\ 4.2072 & -1.0000 & 0.4754\\ 3.3901 & -1.0000 & 0.5900\end{bmatrix}\begin{pmatrix}\tilde{m}_1\\ \tilde{c}_1\\ \tilde{k}_1\end{pmatrix} = \begin{pmatrix}2.9049\\ -1.1843\\ -0.2453\end{pmatrix},$$

and then $\tilde{m}_1 = 0.0945$, $\tilde{c}_1 = 5.7979$, $\tilde{k}_1 = 8.8690$. From the total mass $w = \sum_1^3 m_j = 1.0403$ and the normalized factor $\tilde{w} = \sum_1^3 \tilde{m}_j = 1.4107$ we find that $m_1 = 0.0697$, $m_2 = 0.2332$, $m_3 = 0.7374$, $c_1 = 4.2756$, $c_2 = 3.9105$, $c_3 = 3.8386$, $k_1 = 6.5403$, $k_2 = 6.4281$, and $k_3 = 5.3905$. Therefore, the mass, damping, and stiffness matrices are give by

$$M = \begin{bmatrix} 0.6058 & 0.2332 & 0\\ 0.2332 & 1.9414 & 0.7374\\ 0 & 0.7374 & 1.4749\end{bmatrix}, \quad C = \begin{bmatrix} 8.1861 & -3.9105 & 0\\ -3.9105 & 7.7491 & -3.8386\\ 0 & -3.8386 & 3.8386\end{bmatrix},$$
$$K = \begin{bmatrix} 12.9685 & -6.4281 & 0\\ -6.4281 & 11.8186 & -5.3905\\ 0 & -5.3905 & 5.3905\end{bmatrix}.$$

For another feasible solution $\psi = -8.1870$, by the same way, we constructively obtain the corresponding mass, damping, and stiffness matrices as follows:

$$M = \begin{bmatrix} 1.1353 & 0.1801 & 0\\ 0.1801 & 1.3057 & 0.4727\\ 0 & 0.4727 & 0.9454\end{bmatrix}, \quad C = \begin{bmatrix} 9.0411 & -0.5661 & 0\\ -0.5661 & 3.0267 & -2.4606\\ 0 & -2.4606 & 2.4606\end{bmatrix},$$

$$K = \begin{bmatrix} 13.2695 & -0.6810 & 0 \\ -0.6810 & 4.1364 & -3.4554 \\ 0 & -3.4554 & 3.4554 \end{bmatrix}.$$

From this example, we observe that for each ψ determined by (16), we find a physical realizable solution for Problem A.

3. Problem B

In this section, we consider the solvability of Problem B as in Section 2. Notice that $(\alpha + \beta\mathrm{i}, \mathbf{x}_R + \mathbf{x}_I\mathrm{i})$ and $(\alpha - \beta\mathrm{i}, \mathbf{x}_R - \mathbf{x}_I\mathrm{i})$ are two complex conjugate eigenpairs of the QEP (1), i.e.,

$$\begin{cases} \left((\alpha+\beta\mathrm{i})^2 M + (\alpha+\beta\mathrm{i})C + K\right)(\mathbf{x}_R + \mathbf{x}_I\mathrm{i}) = \mathbf{0}, \\ \left((\alpha-\beta\mathrm{i})^2 M + (\alpha-\beta\mathrm{i})C + K\right)(\mathbf{x}_R - \mathbf{x}_I\mathrm{i}) = \mathbf{0}. \end{cases} \tag{18}$$

Expression (18) can take the following real form:

$$M\begin{bmatrix}\mathbf{x}_R & \mathbf{x}_I\end{bmatrix}\begin{bmatrix}\alpha & \beta \\ -\beta & \alpha\end{bmatrix}^2 + C\begin{bmatrix}\mathbf{x}_R & \mathbf{x}_I\end{bmatrix}\begin{bmatrix}\alpha & \beta \\ -\beta & \alpha\end{bmatrix} + K\begin{bmatrix}\mathbf{x}_R & \mathbf{x}_I\end{bmatrix} = \mathbf{0},$$

or equivalently

$$\begin{cases} M\left[(\alpha^2-\beta^2)\mathbf{x}_R - 2\alpha\beta\mathbf{x}_I\right] + C(\alpha\mathbf{x}_R - \beta\mathbf{x}_I) + K\mathbf{x}_R = \mathbf{0}, \\ M\left[2\alpha\beta\mathbf{x}_R + (\alpha^2-\beta^2)\mathbf{x}_I\right] + C(\beta\mathbf{x}_R + \alpha\mathbf{x}_I) + K\mathbf{x}_I = \mathbf{0}. \end{cases} \tag{19}$$

Recall that for the real eigenpair $(\lambda, \mathbf{x})$, we have

$$(\lambda^2 M + \lambda C + K)\mathbf{x} = \mathbf{0}. \tag{20}$$

Let

$$\begin{cases} \mathbf{x}_R := (x_{1R}, x_{2R}, \ldots, x_{nR})^{\mathrm{T}}, & \mathbf{x}_I := (x_{1I}, x_{2I}, \ldots, x_{nI})^{\mathrm{T}}, \\ (a_{jR})_1^n := x_{j-1,R} + 2x_{jR}, & (a_{jI})_1^n := x_{j-1,I} + 2x_{jI}, \\ (b_{jR})_1^{n-1} := 2x_{jR} + x_{j+1,R}, & (b_{jI})_1^{n-1} := 2x_{jI} + x_{j+1,I}, \\ (d_{jR})_1^n := x_{jR} - x_{j-1,R}, & (d_{jI})_1^n := x_{jI} - x_{j-1,I}, \end{cases} \tag{21}$$

where $x_{0R} = x_{0I} = 0$. Then Eqs. (20) and (19) can be formulated as the following $3n$ pairs of equations

$$\begin{bmatrix} \lambda^2 a_n^x & \lambda d_n^x & d_n^x \\ (\alpha^2-\beta^2)a_{nR} - 2\alpha\beta a_{nI} & \alpha d_{nR} - \beta d_{nI} & d_{nR} \\ 2\alpha\beta a_{nR} + (\alpha^2-\beta^2)a_{nI} & \beta d_{nR} + \alpha d_{nI} & d_{nI} \end{bmatrix}\begin{pmatrix} m_n \\ c_n \\ k_n \end{pmatrix} = \mathbf{0}. \tag{22}$$

and

$$\begin{aligned} &\begin{bmatrix} \lambda_1^2 a_j^{(1)} & \lambda_1 d_j^{(1)} & d_j^{(1)} \\ (\alpha^2-\beta^2)a_{jR} - 2\alpha\beta a_{jI} & \alpha d_{jR} - \beta d_{jI} & d_{jR} \\ 2\alpha\beta a_{jR} + (\alpha^2-\beta^2)a_{jI} & \beta d_{jR} + \alpha d_{jI} & d_{jI} \end{bmatrix}\begin{pmatrix} m_j \\ c_j \\ k_j \end{pmatrix} \\ &\quad = \begin{bmatrix} -\lambda^2 b_j^x & \lambda d_{j+1}^x & d_{j+1}^x \\ -[(\alpha^2-\beta^2)b_{jR} - 2\alpha\beta b_{jI}] & \alpha d_{j+1,R} - \beta d_{j+1,I} & d_{j+1,R} \\ -[2\alpha\beta b_{jR} + (\alpha^2-\beta^2)b_{jI}] & \beta d_{j+1,R} + \alpha d_{j+1,I} & d_{j+1,I} \end{bmatrix}\begin{pmatrix} m_{j+1} \\ c_{j+1} \\ k_{j+1} \end{pmatrix} \end{aligned} \tag{23}$$

for $j = 1, \ldots, n-1$, where a_j^x, b_j^x, and d_j^x are defined as in (9).

Given the total mass $w = \sum_1^n m_j (m_n \neq 0)$, one real eigenvector $\mathbf{x}$, and two complex conjugate eigenpairs $(\alpha + \beta\mathrm{i}, \mathbf{x}_R + \mathbf{x}_R\mathrm{i})$ and $(\alpha - \beta\mathrm{i}, \mathbf{x}_R - \mathbf{x}_I\mathrm{i})$. Let $\tilde{m}_j$, $\tilde{c}_j$, and $\tilde{k}_j$ be defined as in (10). Then we can find $\tilde{c}_n$ and $\tilde{k}_n$ by (22), i.e.,

$$\begin{bmatrix} \lambda d_n^x & d_n^x \\ \alpha d_{nR} - \beta d_{nI} & d_{nR} \\ \beta d_{nR} + \alpha d_{nI} & d_{nI} \end{bmatrix} \begin{pmatrix} \tilde{c}_n \\ \tilde{k}_n \end{pmatrix} = - \begin{pmatrix} \lambda^2 a_n^x \\ (\alpha^2 - \beta^2) a_{nR} - 2\alpha\beta a_{nI} \\ 2\alpha\beta a_{nR} + (\alpha^2 - \beta^2) a_{nI} \end{pmatrix}, \tag{24}$$

where the real number λ is determined by

$$\det \begin{bmatrix} \lambda^2 a_n^x & \lambda d_n^x & d_n^x \\ (\alpha^2 - \beta^2) a_{nR} - 2\alpha\beta a_{nI} & \alpha d_{nR} - \beta d_{nI} & d_{nR} \\ 2\alpha\beta a_{nR} + (\alpha^2 - \beta^2) a_{nI} & \beta d_{nR} + \alpha d_{nI} & d_{nI} \end{bmatrix} = 0 \tag{25}$$

such that

$$\begin{aligned} &\operatorname{rank} \left(\begin{bmatrix} \lambda d_n^x & d_n^x \\ \alpha d_{nR} - \beta d_{nI} & d_{nR} \\ \beta d_{nR} + \alpha d_{nI} & d_{nI} \end{bmatrix} \right) \\ &\quad = \operatorname{rank} \left(\begin{bmatrix} \lambda^2 a_n^x & \lambda d_n^x & d_n^x \\ (\alpha^2 - \beta^2) a_{nR} - 2\alpha\beta a_{nI} & \alpha d_{nR} - \beta d_{nI} & d_{nR} \\ 2\alpha\beta a_{nR} + (\alpha^2 - \beta^2) a_{nI} & \beta d_{nR} + \alpha d_{nI} & d_{nI} \end{bmatrix} \right). \end{aligned}$$

Next, dividing expression (23) by m_n gives rise to

$$\begin{aligned} &\begin{bmatrix} \lambda_1^2 a_j^{(1)} & \lambda_1 d_j^{(1)} & d_j^{(1)} \\ (\alpha^2 - \beta^2) a_{jR} - 2\alpha\beta a_{jI} & \alpha d_{jR} - \beta d_{jI} & d_{jR} \\ 2\alpha\beta a_{jR} + (\alpha^2 - \beta^2) a_{jI} & \beta d_{jR} + \alpha d_{jI} & d_{jI} \end{bmatrix} \begin{pmatrix} \tilde{m}_j \\ \tilde{c}_j \\ \tilde{k}_j \end{pmatrix} \\ &\quad = \begin{bmatrix} -\lambda^2 b_j^x & -\lambda d_{j+1}^x & -d_{j+1}^x \\ -[(\alpha^2 - \beta^2) b_{jR} - 2\alpha\beta b_{jI}] & \alpha d_{j+1,R} - \beta d_{j+1,I} & d_{j+1,R} \\ -[2\alpha\beta b_{jR} + (\alpha^2 - \beta^2) b_{jI}] & \beta d_{j+1,R} + \alpha d_{j+1,I} & d_{j+1,I} \end{bmatrix} \begin{pmatrix} \tilde{m}_{j+1} \\ \tilde{c}_{j+1} \\ \tilde{k}_{j+1} \end{pmatrix} \end{aligned} \tag{26}$$

for $j = n-1, \ldots, 1$. Also, define $\tilde{\mathbf{w}}^{(j)}$ as in (14) and the following notations

$$\mathbf{g}^{(n)} = - \begin{pmatrix} \lambda^2 a_n^x \\ (\alpha^2 - \beta^2) a_{nR} - 2\alpha\beta a_{nI} \\ 2\alpha\beta a_{nR} + (\alpha^2 - \beta^2) a_{nI} \end{pmatrix} \in \mathbb{R}^3,$$

$$A_{nn} = \begin{bmatrix} \lambda d_n^x & d_n^x \\ \alpha d_{nR} - \beta d_{nI} & d_{nR} \\ \beta d_{nR} + \alpha d_{nI} & d_{nI} \end{bmatrix},$$

$$A_{jj} = \begin{bmatrix} \lambda_1^2 a_j^{(1)} & \lambda_1 d_j^{(1)} & d_j^{(1)} \\ (\alpha^2 - \beta^2) a_{jR} - 2\alpha\beta a_{jI} & \alpha d_{jR} - \beta d_{jI} & d_{jR} \\ 2\alpha\beta a_{jR} + (\alpha^2 - \beta^2) a_{jI} & \beta d_{jR} + \alpha d_{jI} & d_{jI} \end{bmatrix} \quad \text{for } 1 \leqslant j \leqslant n-1,$$

and

$$B_{jj}=\begin{bmatrix} -\lambda^2 b_j^x & \lambda d_{j+1}^x & d_{j+1}^x \\ -[(\alpha^2-\beta^2)b_{jR}-2\alpha\beta b_{jI}] & \alpha d_{j+1,R}-\beta d_{j+1,I} & d_{j+1,R} \\ -[2\alpha\beta b_{jR}+(\alpha^2-\beta^2)b_{jI}] & \beta d_{j+1,R}+\alpha d_{j+1,I} & d_{j+1,I}\end{bmatrix}$$

for $1 \leqslant j \leqslant n-1$.

Therefore, we have the following results on the solvability of Problem B.

Theorem 3.1. *Problem B has a nontrivial solution if and only if the following conditions are satisfied*:

(1) *The real number* λ *is determined by* (25) *such that* $\operatorname{rank}(A_{nn}) = \operatorname{rank}([A_{nn}, \mathbf{g}^{(n)}])$;
(2) $\operatorname{rank}(A_{jj}) = \operatorname{rank}([A_{jj}, B_{jj}\tilde{\mathbf{w}}^{(j+1)}])$ *for* $j = n-1, \ldots, 1$.

Corollary 3.2. *Problem B has a unique nontrivial solution if and only if the following conditions are satisfied*:

(1) *The real number* λ *is determined by* (25) *such that* $\operatorname{rank}(A_{nn}) = \operatorname{rank}([A_{nn}, \mathbf{g}^{(n)}]) = 2$;
(2) $\det(A_{jj}) \neq 0$, $(j = n-1, n-2, \ldots, 1)$.

Assume that the given eigendata satisfy the conditions of Theorem 3.1 or Corollary 3.2, one may first get $\tilde{c}_n, \tilde{k}_n$ by (24) and then apply Eqs. (26) successively for $j = n-1, n-2, \ldots, 1$ to determine $(\tilde{m}_j, \tilde{c}_j, \tilde{k}_j)$ for $j = n-1, n-2, \ldots, 1$. Therefore, the solution $(m_j, c_j, k_j)_1^n$ are obtained by $m_j = \tilde{m}_j w/\tilde{w}$, $c_j = \tilde{c}_j w/\tilde{w}$, and $k_j = \tilde{k}_j w/\tilde{w}$, where $\tilde{w} = \sum_1^n \tilde{m}_j$.

To demonstrate our algorithm, an example is presented as follows.

3.1. Example 2

Let $n = 3$ and we randomly generate the total mass w, two complex eigenpairs $(\alpha + \beta\mathrm{i}, \mathbf{x}_R + \mathbf{x}_I\mathrm{i})$ and $(\alpha - \beta\mathrm{i}, \mathbf{x}_R - \mathbf{x}_I\mathrm{i})$ and a real eigenvector $\mathbf{x}$ as follows:

$$\begin{cases} w = 1.4583, \quad \alpha = -0.1732, \quad \beta = 0.7750, \\ \mathbf{x}_R = (0.4428, 0.8551, 0.9752)^{\mathrm{T}}, \\ \mathbf{x}_I = (0.0614, -0.0223, -0.0248)^{\mathrm{T}}, \\ \mathbf{x} = (-0.3662, 0.5406, -0.2788)^{\mathrm{T}}. \end{cases}$$

Then we have

$$\begin{cases} (a_{jR})_1^3 = (0.8856, 2.1531, 2.8055)^{\mathrm{T}}, \\ (b_{jR})_1^2 = (1.7408, 2.6855)^{\mathrm{T}}, \quad (d_{jR})_1^3 = (0.4428, 0.4123, 0.1200)^{\mathrm{T}}, \\ (a_{jI})_1^3 = (0.1227, 0.0168, -0.0719)^{\mathrm{T}}, \\ (b_{jI})_1^2 = (0.1004, -0.0694)^{\mathrm{T}}, \quad (d_{jI})_1^3 = (0.0614, -0.0837, -0.0025)^{\mathrm{T}}, \\ (a_j^x)_1^3 = (-0.7324, 0.7150, -0.0171)^{\mathrm{T}}, \\ (b_j^x)_1^2 = (-0.1918, 0.8024)^{\mathrm{T}}, \quad (d_j^x)_1^3 = (-0.3662, 0.9068, -0.8195)^{\mathrm{T}}. \end{cases}$$

The real number λ is determined by Eq. (25), i.e.,

$$\det\begin{bmatrix}-0.0171\cdot\lambda^2 & -0.8195\cdot\lambda & -0.8195\\ -1.6201 & -0.0188 & 0.1200\\ -0.7120 & 0.0935 & -0.0025\end{bmatrix}=0,$$

which leads to two feasible solutions $\lambda=-1.8498$ or $\lambda=-382.4330$. For $\lambda=-1.8498$, by (24), we have

$$\begin{bmatrix}-0.0188 & 0.1200\\ 0.0935 & -0.0025\end{bmatrix}\begin{pmatrix}\tilde{c}_3\\ \tilde{k}_3\end{pmatrix}=\begin{pmatrix}1.6201\\ 0.7120\end{pmatrix}. \tag{27}$$

We obtain from (27) that $\tilde{c}_3=8.0154$ and $\tilde{k}_3=14.7553$. Setting $j=2$ in Eq. (26) leads to

$$\begin{bmatrix}2.4465 & -1.6774 & 0.9068\\ -1.2241 & -0.0066 & 0.4123\\ -0.5875 & 0.3340 & -0.0837\end{bmatrix}\begin{pmatrix}\tilde{m}_2\\ \tilde{c}_2\\ \tilde{k}_2\end{pmatrix}=\begin{pmatrix}-2.6869\\ 3.1711\\ 1.3932\end{pmatrix},$$

which in turn yields that $\tilde{m}_2=0.5407$, $\tilde{c}_2=7.4800$, $\tilde{k}_2=9.4145$. For $j=1$, Eq. (26) gives rise to

$$\begin{bmatrix}-2.5060 & 0.6774 & -0.3662\\ -0.4724 & -0.1242 & 0.4428\\ -0.3078 & 0.3325 & 0.0614\end{bmatrix}\begin{pmatrix}\tilde{m}_1\\ \tilde{c}_1\\ \tilde{k}_1\end{pmatrix}=\begin{pmatrix}-3.6549\\ 4.3553\\ 1.9944\end{pmatrix},$$

and then $\tilde{m}_1=0.9358$, $\tilde{c}_1=4.6247$, $\tilde{k}_1=12.1316$. From the total mass $w=\sum_1^3 m_j=1.4583$ and the normalized factor $\tilde{w}=\sum_1^3\tilde{m}_j=2.4764$ we find that $m_1=0.5510$, $m_2=0.3184$, $m_3=0.5889$, $c_1=2.7233$, $c_2=4.4047$, $c_3=4.7199$, $k_1=7.1438$, $k_2=5.5438$, and $k_3=8.6888$. Therefore, we get the constructed mass, damping, and stiffness matrices as follows

$$M=\begin{bmatrix}1.7388 & 0.3184 & 0\\ 0.3184 & 1.8145 & 0.5889\\ 0 & 0.5889 & 1.1777\end{bmatrix},\quad C=\begin{bmatrix}7.1280 & -4.4047 & 0\\ -4.4047 & 9.1246 & -4.7199\\ 0 & -4.7199 & 4.7199\end{bmatrix},$$
$$K=\begin{bmatrix}12.6876 & -5.5438 & 0\\ -5.5438 & 14.2326 & -8.6888\\ 0 & -8.6888 & 8.6888\end{bmatrix}.$$

For another feasible solution $\lambda=-382.4330$, we finally obtain the required parameters $m_1=2.0654, m_2=-7.5817, m_3=6.9746; c_1=2.9233, c_2=23.6476, c_3=55.9041; k_1=15.4673, k_2=31.5073, k_3=102.9121$. The corresponding mass, damping, and stiffness matrices are given by

$$M=\begin{bmatrix}-11.033 & -7.5817 & 0\\ -7.5817 & -1.2143 & 6.9746\\ 0 & 6.9746 & 13.949\end{bmatrix},\quad C=\begin{bmatrix}26.571 & -23.648 & 0\\ -23.648 & 79.552 & -55.90\\ 0 & -55.90 & 55.90\end{bmatrix},$$
$$K=\begin{bmatrix}46.97 & -31.51 & 0\\ -31.51 & 134.42 & -102.91\\ 0 & -102.91 & 102.91\end{bmatrix}.$$

From this example, we observe that for $\lambda=-1.8498$, we find a physical model. However, for $\lambda=-382.4330$, the final solution is not physical realistic since $m_2=-7.5817<0$.

4. Problem C

In Sections 2 and 3, we have discussed the solvability of Problems A and B. We note that Problems A and B are special cases of Problem C. So, we can easily develop the sufficient and necessary conditions for the existence of the solution of Problem C.

Given k eigenpairs $\{(\lambda_p, x^{(p)})\}_{p=1}^k$. By (1), we obtain

$$\begin{cases} M[(\alpha_1^2-\beta_1^2)\mathbf{x}_R^{(1)}-2\alpha_1\beta_1\mathbf{x}_I^{(1)}]+C(\alpha_1\mathbf{x}_R^{(1)}-\beta_1\mathbf{x}_I^{(1)})+K\mathbf{x}_R^{(1)}=0,\\ M[2\alpha_1\beta_1\mathbf{x}_R^{(1)}+(\alpha_1^2-\beta_1^2)\mathbf{x}_I^{(1)}]+C(\beta_1\mathbf{x}_R^{(1)}+\alpha_1\mathbf{x}_I^{(1)})+K\mathbf{x}_I^{(1)}=0,\\ \vdots\\ M[(\alpha_\ell^2-\beta_\ell^2)\mathbf{x}_R^{(\ell)}-2\alpha_\ell\beta_\ell\mathbf{x}_I^{(\ell)}]+C(\alpha_\ell\mathbf{x}_R^{(\ell)}-\beta_\ell\mathbf{x}_I^{(\ell)})+K\mathbf{x}_R^{(\ell)}=0,\\ M[2\alpha_\ell\beta_\ell\mathbf{x}_R^{(\ell)}+(\alpha_\ell^2-\beta_\ell^2)\mathbf{x}_I^{(\ell)}]+C(\beta_\ell\mathbf{x}_R^{(\ell)}+\alpha_\ell\mathbf{x}_I^{(\ell)})+K\mathbf{x}_I^{(\ell)}=0,\\ (\lambda_{2\ell+1}^2M+\lambda_{2\ell+1}C+K)\mathbf{x}^{(2\ell+1)}=\mathbf{0},\\ \vdots\\ (\lambda_k^2M+\lambda_kC+K)\mathbf{x}^{(k)}=\mathbf{0}. \end{cases} \tag{28}$$

For $s=1,\ldots,\ell$, let

$$\begin{cases} \mathbf{x}_R^{(s)}:=(x_{1R}^{(s)},x_{2R}^{(s)},\ldots,x_{nR}^{(s)})^{\mathrm{T}}, & \mathbf{x}_I^{(s)}:=(x_{1I}^{(s)},x_{2I}^{(s)},\ldots,x_{nI}^{(s)})^{\mathrm{T}},\\ (a_{jR}^{(s)})_1^n:=x_{j-1,R}^{(s)}+2x_{jR}^{(s)}, & (a_{jI}^{(s)})_1^n:=x_{j-1,I}^{(s)}+2x_{jI}^{(s)},\\ (b_{jR}^{(s)})_1^{n-1}:=2x_{jR}^{(s)}+x_{j+1,R}^{(s)}, & (b_{jI}^{(s)})_1^{n-1}:=2x_{jI}^{(s)}+x_{j+1,I}^{(s)},\\ (d_{jR}^{(s)})_1^n:=x_{jR}^{(s)}-x_{j-1,R}^{(s)}, & (d_{jI}^{(s)})_1^n:=x_{jI}^{(s)}-x_{j-1,I}^{(s)}, \end{cases}$$

where $x_{0R}^{(s)}=x_{0I}^{(s)}=0$. For $s=2\ell+1,\ldots,k$, let

$$(a_j^{(s)})_1^n=x_{j-1}^{(s)}+2x_j^{(s)},\quad (b_j^{(s)})_1^{n-1}=2x_j^{(s)}+x_{j+1}^{(s)},\quad (d_j^{(s)})_1^n=x_j^{(s)}-x_{j-1}^{(s)},$$

where $x_0^{(s)}=0$. Then the linear system (28) can be formulated as the following kn pairs of equations

$$\begin{bmatrix} (\alpha_1^2-\beta_1^2)a_{nR}^{(1)}-2\alpha_1\beta_1a_{nI}^{(1)} & \alpha_1d_{nR}^{(1)}-\beta_1d_{nI}^{(1)} & d_{nR}^{(1)}\\ 2\alpha_1\beta_1a_{nR}^{(1)}+(\alpha_1^2-\beta_1^2)a_{nI}^{(1)} & \beta_1d_{nR}^{(1)}+\alpha_1d_{nI}^{(1)} & d_{nI}^{(1)}\\ \vdots & & \\ (\alpha_\ell^2-\beta_\ell^2)a_{nR}^{(\ell)}-2\alpha_\ell\beta_\ell a_{nI}^{(\ell)} & \alpha_\ell d_{nR}^{(\ell)}-\beta_\ell d_{nI}^{(\ell)} & d_{nR}^{(\ell)}\\ 2\alpha_\ell\beta_\ell a_{nR}^{(\ell)}+(\alpha_\ell^2-\beta_\ell^2)a_{nI}^{(\ell)} & \beta_\ell d_{nR}^{(\ell)}+\alpha_\ell d_{nI}^{(\ell)} & d_{nI}^{(\ell)}\\ \lambda_{2\ell+1}^2a_n^{(2\ell+1)} & \lambda_{2\ell+1}d_n^{(2\ell+1)} & d_n^{(2\ell+1)}\\ \vdots & & \\ \lambda_k^2a_n^{(k)} & \lambda_kd_n^{(k)} & d_n^{(k)} \end{bmatrix}\begin{pmatrix} m_n\\ c_n\\ k_n\end{pmatrix}=0 \tag{29}$$

and

$$
\begin{bmatrix}
(\alpha_1^2-\beta_1^2)a_{jR}^{(1)}-2\alpha_1\beta_1 a_{jI}^{(1)} & \alpha_1 d_{jR}^{(1)}-\beta_1 d_{jI}^{(1)} & d_{jR}^{(1)} \\
2\alpha_1\beta_1 a_{jR}^{(1)}+(\alpha_1^2-\beta_1^2)a_{jI}^{(1)} & \beta_1 d_{jR}^{(1)}+\alpha_1 d_{jI}^{(1)} & d_{jI}^{(1)} \\
\vdots & & \\
(\alpha_\ell^2-\beta_\ell^2)a_{jR}^{(\ell)}-2\alpha_\ell\beta_\ell a_{jI}^{(\ell)} & \alpha_\ell d_{jR}^{(\ell)}-\beta_\ell d_{jI}^{(\ell)} & d_{jR}^{(\ell)} \\
2\alpha_\ell\beta_\ell a_{jR}^{(\ell)}+(\alpha_\ell^2-\beta_\ell^2)a_{jI}^{(\ell)} & \beta_\ell d_{jR}^{(\ell)}+\alpha_\ell d_{jI}^{(\ell)} & d_{jI}^{(\ell)} \\
\lambda_{2\ell+1}^2 a_j^{(2\ell+1)} & \lambda_{2\ell+1} d_j^{(2\ell+1)} & d_j^{(2\ell+1)} \\
\vdots & & \\
\lambda_k^2 a_j^{(k)} & \lambda_k d_j^{(k)} & d_j^{(k)}
\end{bmatrix}
\begin{pmatrix} m_j \\ c_j \\ k_j \end{pmatrix}
$$

$$
= \begin{bmatrix}
-[(\alpha_1^2-\beta_1^2)b_{jR}^{(1)}-2\alpha_1\beta_1 b_{jI}^{(1)}] & \alpha_1 d_{j+1,R}^{(1)}-\beta_1 d_{j+1,I}^{(1)} & d_{j+1,R}^{(1)} \\
-[2\alpha_1\beta_1 b_{jR}^{(1)}+(\alpha_1^2-\beta_1^2)b_{jI}^{(1)}] & \beta_1 d_{j+1,R}^{(1)}+\alpha_1 d_{j+1,I}^{(1)} & d_{j+1,I}^{(1)} \\
\vdots & & \\
-[(\alpha_\ell^2-\beta_\ell^2)b_{jR}^{(\ell)}-2\alpha_\ell\beta_\ell b_{jI}^{(\ell)}] & \alpha_\ell d_{j+1,R}^{(\ell)}-\beta_\ell d_{j+1,I}^{(\ell)} & d_{j+1,R}^{(\ell)} \\
-[2\alpha_\ell\beta_\ell b_{jR}^{(\ell)}+(\alpha_\ell^2-\beta_\ell^2)b_{jI}^{(\ell)}] & \beta_\ell d_{j+1,R}^{(\ell)}+\alpha_\ell d_{j+1,I}^{(\ell)} & d_{j+1,I}^{(\ell)} \\
-\lambda_{2\ell+1}^2 b_j^{(2\ell+1)} & \lambda_{2\ell+1} d_{j+1}^{(2\ell+1)} & d_{j+1}^{(2\ell+1)} \\
\vdots & & \\
-\lambda_k^2 b_j^{(k)} & \lambda_k d_{j+1}^{(k)} & d_{j+1}^{(k)}
\end{bmatrix}
\begin{pmatrix} m_{j+1} \\ c_{j+1} \\ k_{j+1} \end{pmatrix}
\tag{30}
$$

for $j = 1, \ldots, n-1$.

Given the total mass $w = \sum_1^n m_j$. Let $\tilde{m}_j$, $\tilde{c}_j$, and $\tilde{k}_j$ be defined as in (10). Then, by (29), we can find $\tilde{c}_n$ and $\tilde{k}_n$, i.e.,

$$
\begin{bmatrix}
\alpha_1 d_{nR}^{(1)}-\beta_1 d_{nI}^{(1)} & d_{nR}^{(1)} \\
\beta_1 d_{nR}^{(1)}+\alpha_1 d_{nI}^{(1)} & d_{nI}^{(1)} \\
\vdots & \\
\alpha_\ell d_{nR}^{(\ell)}-\beta_\ell d_{nI}^{(\ell)} & d_{nR}^{(\ell)} \\
\beta_\ell d_{nR}^{(\ell)}+\alpha_\ell d_{nI}^{(\ell)} & d_{nI}^{(\ell)} \\
\lambda_{2\ell+1} d_n^{(2\ell+1)} & d_n^{(2\ell+1)} \\
\vdots & \\
\lambda_k d_n^{(k)} & d_n^{(k)}
\end{bmatrix}
\begin{pmatrix} \tilde{c}_n \\ \tilde{k}_n \end{pmatrix}
= -\begin{pmatrix}
(\alpha_1^2-\beta_1^2)a_{nR}^{(1)}-2\alpha_1\beta_1 a_{nI}^{(1)} \\
2\alpha_1\beta_1 a_{nR}^{(1)}+(\alpha_1^2-\beta_1^2)a_{nI}^{(1)} \\
\vdots \\
(\alpha_\ell^2-\beta_\ell^2)a_{nR}^{(\ell)}-2\alpha_\ell\beta_\ell a_{nI}^{(\ell)} \\
2\alpha_\ell\beta_\ell a_{nR}^{(\ell)}+(\alpha_\ell^2-\beta_\ell^2)a_{nI}^{(\ell)} \\
\lambda_{2\ell+1}^2 a_n^{(2\ell+1)} \\
\vdots \\
\lambda_k^2 a_n^{(k)}
\end{pmatrix}. \tag{31}
$$

Next, dividing expression (30) by m_n gives rise to

$$\begin{bmatrix} (\alpha_1^2-\beta_1^2)a_{jR}^{(1)}-2\alpha_1\beta_1 a_{jI}^{(1)} & \alpha_1 d_{jR}^{(1)}-\beta_1 d_{jI}^{(1)} & d_{jR}^{(1)} \\ 2\alpha_1\beta_1 a_{jR}^{(1)}+(\alpha_1^2-\beta_1^2)a_{jI}^{(1)} & \beta_1 d_{jR}^{(1)}+\alpha_1 d_{jI}^{(1)} & d_{jI}^{(1)} \\ \vdots & & \\ (\alpha_\ell^2-\beta_\ell^2)a_{jR}^{(\ell)}-2\alpha_\ell\beta_\ell a_{jI}^{(\ell)} & \alpha_\ell d_{jR}^{(\ell)}-\beta_\ell d_{jI}^{(\ell)} & d_{jR}^{(\ell)} \\ 2\alpha_\ell\beta_\ell a_{jR}^{(\ell)}+(\alpha_\ell^2-\beta_\ell^2)a_{jI}^{(\ell)} & \beta_\ell d_{jR}^{(\ell)}+\alpha_\ell d_{jI}^{(\ell)} & d_{jI}^{(\ell)} \\ \lambda_{2\ell+1}^2 a_j^{(2\ell+1)} & \lambda_{2\ell+1} d_j^{(2\ell+1)} & d_j^{(2\ell+1)} \\ \vdots & & \\ \lambda_k^2 a_j^{(k)} & \lambda_k d_j^{(k)} & d_j^{(k)} \end{bmatrix} \begin{pmatrix} \tilde{m}_j \\ \tilde{c}_j \\ \tilde{k}_j \end{pmatrix}$$

$$= \begin{bmatrix} -[(\alpha_1^2-\beta_1^2)b_{jR}^{(1)}-2\alpha_1\beta_1 b_{jI}^{(1)}] & \alpha_1 d_{j+1,R}^{(1)}-\beta_1 d_{j+1,I}^{(1)} & d_{j+1,R}^{(1)} \\ -[2\alpha_1\beta_1 b_{jR}^{(1)}+(\alpha_1^2-\beta_1^2)b_{jI}^{(1)}] & \beta_1 d_{j+1,R}^{(1)}+\alpha_1 d_{j+1,I}^{(1)} & d_{j+1,I}^{(1)} \\ \vdots & & \\ -[(\alpha_\ell^2-\beta_\ell^2)b_{jR}^{(\ell)}-2\alpha_\ell\beta_\ell b_{jI}^{(\ell)}] & \alpha_\ell d_{j+1,R}^{(\ell)}-\beta_\ell d_{j+1,I}^{(\ell)} & d_{j+1,R}^{(\ell)} \\ -[2\alpha_\ell\beta_\ell b_{jR}^{(\ell)}+(\alpha_\ell^2-\beta_\ell^2)b_{jI}^{(\ell)}] & \beta_\ell d_{j+1,R}^{(\ell)}+\alpha_\ell d_{j+1,I}^{(\ell)} & d_{j+1,I}^{(\ell)} \\ -\lambda_{2\ell+1}^2 b_j^{(2\ell+1)} & \lambda_{2\ell+1} d_{j+1}^{(2\ell+1)} & d_{j+1}^{(2\ell+1)} \\ \vdots & & \\ -\lambda_k^2 b_j^{(k)} & \lambda_k d_{j+1}^{(k)} & d_{j+1}^{(k)} \end{bmatrix} \begin{pmatrix} \tilde{m}_{j+1} \\ \tilde{c}_{j+1} \\ \tilde{k}_{j+1} \end{pmatrix} \tag{32}$$

for $j = 1, \ldots, n-1$. Also, define the following notations with $\tilde{\mathbf{w}}^{(j)}$ defined as in (14) and

$$\mathbf{g}^{(n)} = -\begin{pmatrix} (\alpha_1^2-\beta_1^2)a_{nR}^{(1)}-2\alpha_1\beta_1 a_{nI}^{(1)} \\ 2\alpha_1\beta_1 a_{nR}^{(1)}+(\alpha_1^2-\beta_1^2)a_{nI}^{(1)} \\ \vdots \\ (\alpha_\ell^2-\beta_\ell^2)a_{nR}^{(\ell)}-2\alpha_\ell\beta_\ell a_{nI}^{(\ell)} \\ 2\alpha_\ell\beta_\ell a_{nR}^{(\ell)}+(\alpha_\ell^2-\beta_\ell^2)a_{nI}^{(\ell)} \\ \lambda_{2\ell+1}^2 a_n^{(2\ell+1)} \\ \vdots \\ \lambda_k^2 a_n^{(k)} \end{pmatrix} \in \mathbb{R}^k, \quad A_{nn} = \begin{bmatrix} \alpha_1 d_{nR}^{(1)}-\beta_1 d_{nI}^{(1)} & d_{nR}^{(1)} \\ \beta_1 d_{nR}^{(1)}+\alpha_1 d_{nI}^{(1)} & d_{nI}^{(1)} \\ \vdots & \\ \alpha_\ell d_{nR}^{(\ell)}-\beta_\ell d_{nI}^{(\ell)} & d_{nR}^{(\ell)} \\ \beta_\ell d_{nR}^{(\ell)}+\alpha_\ell d_{nI}^{(\ell)} & d_{nI}^{(\ell)} \\ \lambda_{2\ell+1} d_n^{(2\ell+1)} & d_n^{(2\ell+1)} \\ \vdots & \\ \lambda_k d_n^{(k)} & d_n^{(k)} \end{bmatrix}, \tag{33}$$

$$A_{jj} = \begin{bmatrix} (\alpha_1^2-\beta_1^2)a_{jR}^{(1)} - 2\alpha_1\beta_1 a_{jI}^{(1)} & \alpha_1 d_{jR}^{(1)} - \beta_1 d_{jI}^{(1)} & d_{jR}^{(1)} \\ 2\alpha_1\beta_1 a_{jR}^{(1)} + (\alpha_1^2-\beta_1^2)a_{jI}^{(1)} & \beta_1 d_{jR}^{(1)} + \alpha_1 d_{jI}^{(1)} & d_{jI}^{(1)} \\ \vdots & & \\ (\alpha_\ell^2-\beta_\ell^2)a_{jR}^{(\ell)} - 2\alpha_\ell\beta_\ell a_{jI}^{(\ell)} & \alpha_\ell d_{jR}^{(\ell)} - \beta_\ell d_{jI}^{(\ell)} & d_{jR}^{(\ell)} \\ 2\alpha_\ell\beta_\ell a_{jR}^{(\ell)} + (\alpha_\ell^2-\beta_\ell^2)a_{jI}^{(\ell)} & \beta_\ell d_{jR}^{(\ell)} + \alpha_\ell d_{jI}^{(\ell)} & d_{jI}^{(\ell)} \\ \lambda_{2\ell+1}^2 a_j^{(2\ell+1)} & \lambda_{2\ell+1} d_j^{(2\ell+1)} & d_j^{(2\ell+1)} \\ \vdots & & \\ \lambda_k^2 a_j^{(k)} & \lambda_k d_j^{(k)} & d_j^{(k)} \end{bmatrix} \quad \text{for } 1 \leqslant j \leqslant n-1, \tag{34}$$

and

$$B_{jj} = \begin{bmatrix} -[(\alpha_1^2-\beta_1^2)b_{jR}^{(1)} - 2\alpha_1\beta_1 b_{jI}^{(1)}] & \alpha_1 d_{j+1,R}^{(1)} - \beta_1 d_{j+1,I}^{(1)} & d_{j+1,R}^{(1)} \\ -[2\alpha_1\beta_1 b_{jR}^{(1)} + (\alpha_1^2-\beta_1^2)b_{jI}^{(1)}] & \beta_1 d_{j+1,R}^{(1)} + \alpha_1 d_{j+1,I}^{(1)} & d_{j+1,I}^{(1)} \\ \vdots & & \\ -[(\alpha_\ell^2-\beta_\ell^2)b_{jR}^{(\ell)} - 2\alpha_\ell\beta_\ell b_{jI}^{(\ell)}] & \alpha_\ell d_{j+1,R}^{(\ell)} - \beta_\ell d_{j+1,I}^{(\ell)} & d_{j+1,R}^{(\ell)} \\ -[2\alpha_\ell\beta_\ell b_{jR}^{(\ell)} + (\alpha_\ell^2-\beta_\ell^2)b_{jI}^{(\ell)}] & \beta_\ell d_{j+1,R}^{(\ell)} + \alpha_\ell d_{j+1,I}^{(\ell)} & d_{j+1,I}^{(\ell)} \\ -\lambda_{2\ell+1}^2 b_j^{(2\ell+1)} & \lambda_{2\ell+1} d_{j+1}^{(2\ell+1)} & d_{j+1}^{(2\ell+1)} \\ \vdots & & \\ -\lambda_k^2 b_j^{(k)} & \lambda_k d_{j+1}^{(k)} & d_{j+1}^{(k)} \end{bmatrix}$$

$$\text{for } 1 \leqslant j \leqslant n-1, \tag{35}$$

Therefore, we have the following results on the solvability of Problem C.

Theorem 4.1. *Problem* C *has a nontrivial solution if and only if the following conditions are satisfied*:

(1) $\operatorname{rank}(A_{nn}) = \operatorname{rank}([A_{nn}, \mathbf{g}^{(n)}])$;
(2) $\operatorname{rank}(A_{jj}) = \operatorname{rank}([A_{jj}, B_{jj}\tilde{\mathbf{w}}^{(j+1)}])$ *for* $j = n-1, n-2, \ldots, 1$.

Corollary 4.2. *Problem* C *has a unique nontrivial solution if and only if the following conditions are satisfied*:

(1) $\operatorname{rank}(A_{nn}) = \operatorname{rank}([A_{nn}, \mathbf{g}^{(n)}]) = 2$;
(2) $\det(A_{jj}) \neq 0$ $(j = n-1, n-2, \ldots, 1)$.

Remark 4.3. We observe that Problem C is under-determined when $k < 3$ and over-determined when $k \geqslant 3$. When Problem C is under-determined, it follows that there infinite solutions if the conditions in Theorem 4.1 are satisfied. When Problem C is over-determined, under the conditions in Theorem 4.1 or Corollary 4.2, one may find some of parameters $(m_j, c_j, k_j)_1^n$ are not positive. It seems difficult to find the sufficient and necessary conditions on the eigendata so that we can construct a physical feasible solution.

In particular, in the following subsections, we will establish the sufficient and necessary conditions on given two real or complex conjugate eigenpairs so that there exists a physical solution to the under-determined problem.

4.1. Real eigenpairs

Given the total mass $w = \sum_1^n m_j$ and two real eigenpairs $(\lambda, \mathbf{x})$ and $(\phi, \mathbf{y})$. Then we obtain the following system

$$\begin{cases}(\lambda^2 M + \lambda C + K)\mathbf{x} = \mathbf{0},\\ (\phi^2 M + \phi C + K)\mathbf{y} = \mathbf{0}.\end{cases} \tag{36}$$

We can rewrite (36) as the following $2n$ pairs of equations:

$$\begin{cases}\lambda^2 a_n^x m_n + \lambda d_n^x c_n + d_n^x k_n = 0,\\ \phi^2 a_n^y m_n + \phi d_n^y c_n + d_n^y k_n = 0,\end{cases} \tag{37}$$

and

$$\begin{cases}\lambda^2 a_j^x m_j + \lambda d_j^x c_j + d_j^x k_j + \lambda^2 b_j^x m_{j+1} - \lambda d_{j+1}^x c_{j+1} - d_{j+1}^x k_{j+1} = 0,\\ \phi^2 a_j^y m_j + \phi d_j^y c_j + d_j^y k_j + \phi^2 b_j^y m_{j+1} - \phi d_{j+1}^y c_{j+1} - d_{j+1}^y k_{j+1} = 0\end{cases} \tag{38}$$

for $j = n-1, \ldots, 1$, where a_j^x, b_j^x, d_j^x and a_j^y, b_j^y, d_j^y are defined in (9).

For $j = 1, \ldots, n$, define

$$\Phi_j := \begin{bmatrix}\lambda d_j^x & d_j^x\\ \phi d_j^y & d_j^y\end{bmatrix}, \quad \mathbf{u}_j := -\begin{pmatrix}\lambda^2 a_j^x\\ \phi^2 a_j^y\end{pmatrix}, \quad \mathbf{v}_j := -\begin{pmatrix}\lambda^2 b_j^x\\ \phi^2 b_j^y\end{pmatrix}, \tag{39}$$

where $\mathbf{v}_n = \mathbf{0} \in \mathbb{R}^2$. Let $\tilde{m}_j$, $\tilde{c}_j$, and $\tilde{k}_j$ be defined as in (10). Then we have from (37) and (38) that

$$\Phi_n \begin{pmatrix}\tilde{c}_n\\ \tilde{k}_n\end{pmatrix} = \mathbf{u}_n, \tag{40}$$

and

$$\Phi_j \begin{pmatrix}\tilde{c}_j\\ \tilde{k}_j\end{pmatrix} = \Phi_{j+1}\begin{pmatrix}\tilde{c}_{j+1}\\ \tilde{k}_{j+1}\end{pmatrix} + \tilde{m}_j \mathbf{u}_j + \tilde{m}_{j+1}\mathbf{v}_j \tag{41}$$

for $j = n-1, \ldots, 1$.

Suppose that all matrices Φ_j are nonsingular, then the inverse of Φ_j is expressed explicitly as

$$\Phi_j^{-1} = \frac{1}{(\lambda - \phi) d_j^x d_j^y}\begin{bmatrix} d_j^y & -d_j^x\\ -\phi d_j^y & \lambda d_j^x\end{bmatrix}.$$

Then, we obtain the equivalent equations

$$\begin{pmatrix} \tilde{c}_n \\ \tilde{k}_n \end{pmatrix} = \Phi_n^{-1}\mathbf{u}_n,$$

and

$$\begin{pmatrix} \tilde{c}_j \\ \tilde{k}_j \end{pmatrix} = \Phi_j^{-1}[(\mathbf{u}_n + \mathbf{v}_{n-1}) + \tilde{m}_{n-1}(\mathbf{u}_{n-1} + \mathbf{v}_{n-2}) + \cdots + \tilde{m}_j\mathbf{u}_j]$$

for $j = n-1, \ldots, 1$.

We note that the entries of Φ_j^{-1}, $\mathbf{u}_j$, and $\mathbf{v}_j$ are known in terms of the given eigendata. Define

$$\Phi_s^{-1}\mathbf{u}_t = \begin{pmatrix} a_{st} \\ b_{st} \end{pmatrix} \quad \text{and} \quad \Phi_s^{-1}\mathbf{v}_t = \begin{pmatrix} c_{st} \\ d_{st} \end{pmatrix}, \quad s, t = 1, \ldots, n, \tag{42}$$

where $c_{sn} = d_{sn} = 0$ for $s = 1, \ldots, n$.

Let the $(n-1) \times (n-1)$ upper triangular matrices A, B and the $(n-1)$-vectors $\mathbf{a}$ and $\mathbf{b}$ be defined as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} + c_{11} & \cdots & a_{1,n-1} + c_{1,n-2} \\ 0 & a_{22} & \cdots & a_{2,n-1} + c_{2,n-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} \end{bmatrix},$$

$$B = \begin{bmatrix} b_{11} & b_{12} + d_{11} & \cdots & b_{1,n-1} + d_{1,n-2} \\ 0 & b_{22} & \cdots & b_{2,n-1} + d_{2,n-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_{n-1,n-1} \end{bmatrix},$$

and

$$\mathbf{a} = \begin{pmatrix} a_{1,n} + c_{1,n-1} \\ a_{2,n} + c_{2,n-1} \\ \vdots \\ a_{n-1,n} + c_{n-1,n-1} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_{1,n} + d_{1,n-1} \\ b_{2,n} + d_{2,n-1} \\ \vdots \\ b_{n-1,n} + d_{n-1,n-1} \end{pmatrix}.$$

From (40) and (41), we get the following equivalent conditions.

Theorem 4.4. *Given the total mass $w = \sum_1^n m_j$ and two real eigenpairs $(\lambda, \mathbf{x})$ and $(\phi, \mathbf{y})$. Suppose that all matrices Φ_j defined in (39) are nonsingular. Then finding the positive physical parameters $\{c_j\}_1^n$ and $\{k_j\}_1^n$ in terms of the positive $\{m_j\}_1^n$ is equivalent to showing that*

$$a_{nn} > 0, \quad b_{nn} > 0, \tag{43}$$

and the system of inequalities

$$\begin{bmatrix} A \\ B \end{bmatrix} \tilde{\mathbf{m}}(1:n-1) + \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} > 0 \tag{44}$$

is consistent for some $\tilde{\mathbf{m}}(1:n-1) = (\tilde{m}_1, \ldots, \tilde{m}_{n-1})^{\mathrm{T}}$.

4.2. Complex conjugate eigenpairs

Given the total mass $w = \sum_1^n m_j$ and two complex conjugate eigenpairs $(\alpha + \beta \mathrm{i}, \mathbf{x}_R + \mathbf{x}_I \mathrm{i})$ and $(\alpha - \beta \mathrm{i}, \mathbf{x}_R - \mathbf{x}_I \mathrm{i})$. Then we have the system (19), which, in turn, can be rewritten as the following $2n$ pairs of equations:

$$\begin{bmatrix} (\alpha^2-\beta^2)a_{nR} - 2\alpha\beta a_{nI} & \alpha d_{nR} - \beta d_{nI} & d_{nR} \\ 2\alpha\beta a_{nR} + (\alpha^2-\beta^2)a_{nI} & \beta d_{nR} + \alpha d_{nI} & d_{nI} \end{bmatrix} \begin{pmatrix} m_n \\ c_n \\ k_n \end{pmatrix} = \mathbf{0}. \tag{45}$$

and

$$\begin{aligned} &\begin{bmatrix} (\alpha^2-\beta^2)a_{jR} - 2\alpha\beta a_{jI} & \alpha d_{jR} - \beta d_{jI} & d_{jR} \\ 2\alpha\beta a_{jR} + (\alpha^2-\beta^2)a_{jI} & \beta d_{jR} + \alpha d_{jI} & d_{jI} \end{bmatrix} \begin{pmatrix} m_j \\ c_j \\ k_j \end{pmatrix} \\ &\quad = \begin{bmatrix} -[(\alpha^2-\beta^2)b_{jR} - 2\alpha\beta b_{jI}] & \alpha d_{j+1,R} - \beta d_{j+1,I} & d_{j+1,R} \\ -[2\alpha\beta b_{jR} + (\alpha^2-\beta^2)b_{jI}] & \beta d_{j+1,R} + \alpha d_{j+1,I} & d_{j+1,I} \end{bmatrix} \begin{pmatrix} m_{j+1} \\ c_{j+1} \\ k_{j+1} \end{pmatrix} \end{aligned} \tag{46}$$

for $j = 1, \ldots, n-1$, where $a_{jR}, a_{jI}, b_{jR}, b_{jI}, d_{jR}$, and d_{jI} are defined as in (21).

For $j = 1, \ldots, n$, define

$$\Psi_j := \begin{bmatrix} \alpha d_{jR} - \beta d_{jI} & d_{jR} \\ \beta d_{jR} + \alpha d_{jI} & d_{jI} \end{bmatrix}, \tag{47}$$

and

$$\mathbf{u}_j^c := -\begin{pmatrix} (\alpha^2-\beta^2)a_{jR} - 2\alpha\beta a_{jI} \\ 2\alpha\beta a_{jR} + (\alpha^2-\beta^2)a_{jI} \end{pmatrix}, \quad \mathbf{v}_j^c := -\begin{pmatrix} (\alpha^2-\beta^2)b_{jR} - 2\alpha\beta b_{jI} \\ 2\alpha\beta b_{jR} + (\alpha^2-\beta^2)b_{jI} \end{pmatrix}, \tag{48}$$

where $\mathbf{v}_n = \mathbf{0} \in \mathbb{R}^2$. Let $\tilde{m}_j$, $\tilde{c}_j$, and $\tilde{k}_j$ be defined as in (10). Then we have from (45) and (46) that

$$\Psi_n \begin{pmatrix} \tilde{c}_n \\ \tilde{k}_n \end{pmatrix} = \mathbf{u}_n^c, \tag{49}$$

and

$$\Psi_j \begin{pmatrix} \tilde{c}_j \\ \tilde{k}_j \end{pmatrix} = \Psi_{j+1} \begin{pmatrix} \tilde{c}_{j+1} \\ \tilde{k}_{j+1} \end{pmatrix} + \tilde{m}_j \mathbf{u}_j^c + \tilde{m}_{j+1} \mathbf{v}_j^c \tag{50}$$

for $j = n-1, \ldots, 1$.

Suppose that all matrices Ψ_j are nonsingular, then the inverse of Ψ_j is expressed explicitly as

$$\Psi_j^{-1} = -\frac{1}{\beta(d_{jR}^2 + d_{jI}^2)} \begin{bmatrix} d_{jI} & -d_{jR} \\ -(\beta d_{jR} + \alpha d_{jI}) & \alpha d_{jR} - \beta d_{jI} \end{bmatrix}.$$

Then, we obtain the equivalent equations

$$\begin{pmatrix} \tilde{c}_n \\ \tilde{k}_n \end{pmatrix} = \Psi_n^{-1} \mathbf{u}_n^c,$$

and

$$\begin{pmatrix} \tilde{c}_j \\ \tilde{k}_j \end{pmatrix} = \Psi_j^{-1} [(\mathbf{u}_n^c + \mathbf{v}_{n-1}^c) + \tilde{m}_{n-1}(\mathbf{u}_{n-1}^c + \mathbf{v}_{n-2}^c) + \cdots + \tilde{m}_j \mathbf{u}_j^c]$$

for $j = n-1, \ldots, 1$.

We note that the entries of Ψ_j^{-1}, $\mathbf{u}_j^c$, and $\mathbf{v}_j^c$ are known in terms of the given eigendata. Define

$$\Psi_s^{-1}\mathbf{u}_t^c = \begin{pmatrix} a_{st}^c \\ b_{st}^c \end{pmatrix} \quad \text{and} \quad \Phi_s^{-1}\mathbf{v}_t^c = \begin{pmatrix} c_{st}^c \\ d_{st}^c \end{pmatrix}, \quad s, t = 1, \ldots, n, \tag{51}$$

where $c_{sn}^c = d_{sn}^c = 0$ for $s = 1, \ldots, n$.

Let the $(n-1) \times (n-1)$ upper triangular matrices A^c, B^c and the $(n-1)$-vectors $\mathbf{a}^c$ and $\mathbf{b}^c$ be defined as follows:

$$A^c = \begin{bmatrix} a_{11}^c & a_{12}^c + c_{11}^c & \cdots & a_{1,n-1}^c + c_{1,n-2}^c \\ 0 & a_{22}^c & \cdots & a_{2,n-1}^c + c_{2,n-2}^c \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1}^c \end{bmatrix},$$

$$B^c = \begin{bmatrix} b_{11} & b_{12}^c + d_{11}^c & \cdots & b_{1,n-1}^c + d_{1,n-2}^c \\ 0 & b_{22}^c & \cdots & b_{2,n-1}^c + d_{2,n-2}^c \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_{n-1,n-1}^c \end{bmatrix},$$

and

$$\mathbf{a}^c = \begin{pmatrix} a_{1,n}^c + c_{1,n-1}^c \\ a_{2,n}^c + c_{2,n-1}^c \\ \vdots \\ a_{n-1,n}^c + c_{n-1,n-1}^c \end{pmatrix}, \quad \mathbf{b}^c = \begin{pmatrix} b_{1,n}^c + d_{1,n-1}^c \\ b_{2,n}^c + d_{2,n-1}^c \\ \vdots \\ b_{n-1,n} + d_{n-1,n-1}^c \end{pmatrix}.$$

From (49) and (50), we have the following solvability conditions.

Theorem 4.5. *Given the total mass $w = \sum_1^n m_j$ and two complex conjugate eigenpairs $(\alpha + \beta\mathrm{i}, \mathbf{x}_R + \mathbf{x}_I\mathrm{i})$ and $(\alpha - \beta\mathrm{i}, \mathbf{x}_R - \mathbf{x}_I\mathrm{i})$. Suppose that all matrices Ψ_j defined in (47) are nonsingular. Then finding the positive physical parameters $\{c_j\}_1^n$ and $\{k_j\}_1^n$ in terms of the positive $\{m_j\}_1^n$ is equivalent to showing that*

$$a_{nn}^c > 0, \quad b_{nn}^c > 0, \tag{52}$$

and the system of inequalities

$$\begin{bmatrix} A^c \\ B^c \end{bmatrix} \widetilde{\mathbf{m}}(1:n-1) + \begin{pmatrix} \mathbf{a}^c \\ \mathbf{b}^c \end{pmatrix} > 0 \tag{53}$$

is consistent for some $\tilde{\mathbf{m}}(1:n-1) = (\tilde{m}_1, \ldots, \tilde{m}_{n-1})^{\mathrm{T}}$.

4.3. Numerical algorithm

Suppose the total mass and two real or complex conjugate eigenpairs are given, we have shown that the solvability of Problem C is equivalently converted to the solution of the system of strict inequalities. In this subsection, we present a numerical procedure for finding the positive physical parameters $\{c_j\}_1^n$ and $\{k_j\}_1^n$ (if exists) in terms of positive parameters $\{m_j\}_1^n$ given the total mass w and two arbitrary real or complex conjugate eigenpairs. Suppose that we have an estimate of the analytic model $P_o(\lambda) := \lambda^2 M_o + \lambda C_o + K_o$ with the corresponding analytic parameters $\{m_j^o\}_1^n$,

$\{c_j^o\}_1^n$ and $\{k_j^o\}_1^n$. To preserve the structural connectivity, it is natural to consider the optimal parameter updating problem:

$$\begin{aligned}
&\min && \tfrac{1}{2}\|\widetilde{\mathbf{m}}(1:n-1)-\widetilde{\mathbf{m}}^o(1:n-1)\|^2\\
&\text{subject to} && \begin{bmatrix} A\\ B\end{bmatrix}\widetilde{\mathbf{m}}(1:n-1)+\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\geqslant 0,\\
& && \left(\text{or} \begin{bmatrix} A^c\\ B^c\end{bmatrix}\widetilde{\mathbf{m}}(1:n-1)+\begin{pmatrix}\mathbf{a}^c\\ \mathbf{b}^c\end{pmatrix}\geqslant 0,\right)\\
& && \widetilde{\mathbf{m}}(1:n-1)\geqslant 0.
\end{aligned} \tag{54}$$

This is a constrained linear least-squares problem, which can be solved by the `Matlab` routine `lsqlin` based on an active set method [21,5]).

Given the total mass $w=\sum_1^n m_j$ and two real eigenpairs $(\lambda,\mathbf{x})$ and $(\phi,\mathbf{y})$ (or two complex conjugate eigenpairs $(\alpha+\beta\mathrm{i},\mathbf{x}_R+\mathbf{x}_I\mathrm{i})$ and $(\alpha-\beta\mathrm{i},\mathbf{x}_R-\mathbf{x}_I\mathrm{i})$). Let $\{m_j^o\}_1^n$, $\{c_j^o\}_1^n$, and $\{k_j^o\}_1^n$ be the estimate of the analytic parameters. Based on the above analysis, we can construct the mass, damping, and stiffness parameters $\{m_j\}_1^n$, $\{c_j\}_1^n$, and $\{k_j\}_1^n$ for the pencil $\lambda^2M+\lambda C+K$, which has the given eigenpairs or show that such parameters do not exist.

Algorithm 4.6. (1) For $j=1,\ldots,n$, compute the scalars a_j^x, b_j^x, d_j^x and a_j^y, b_j^y, d_j^y by (9) (or $a_{jR}, a_{jI}, b_{jR}, b_{jI}, d_{jR}$, and d_{jI} by (21)).

(2) For $j=1,\ldots,n$, compute the vectors $\mathbf{u}_j$ and $\mathbf{v}_j$ by (39) (or $\mathbf{u}_j^c$ and $\mathbf{v}_j^c$ by (48)).

(3) Check the nonsingularity of Φ_j (or Ψ_j) for all $j=1,\ldots,n$. Otherwise, stop and report the given data is not generic.

(4) Check the condition (43) (or (52)) is satisfied. Otherwise, stop and report the given data is not generic.

(5) Form the upper triangular matrices A and B by (42) (or A^c and B^c by (51)).

(6) Solve the quadratic programming (54).

(a) If there exists a solution $\tilde{\mathbf{m}}(1:n-1)>0$ satisfying (44) (or (53)), then define $\tilde{\mathbf{c}}(1:n-1)=(\tilde{c}_1,\ldots,\tilde{c}_{n-1})^{\mathrm{T}} = A\tilde{m}+\mathbf{a}$ and $\tilde{c}_n=a_{nn}$ and $\tilde{\mathbf{k}}(1:n-1):=(\tilde{k}_1,\ldots,\tilde{k}_{n-1})^{\mathrm{T}}=B\tilde{m}+\mathbf{b}$ and $\tilde{k}_n=b_{nn}$ (or $\tilde{\mathbf{c}}(1:n-1)=A^c\tilde{m}+\mathbf{a}^c$ and $\tilde{c}_n=a_{nn}^c$ and $\tilde{\mathbf{k}}(1:n-1)=B^c\tilde{m}+\mathbf{b}^c$ and $\tilde{k}_n=b_{nn}^c$).

(b) If there is no a solution $\tilde{\mathbf{m}}(1:n-1)>0$ satisfying (44) (or (53)), then the given eigendata are infeasible.

(7) Compute the physical mass, damping, and stiffness parameters $\{m_j\}_1^n$, $\{c_j\}_1^n$, and $\{k_j\}_1^n$ by (15).

We can observe the behavior of Algorithm 4.6 from the later numerical results.

5. Reconstruction from noisy data

In practice, we often use experimentally measured data as the given eigendata. It is very known that all practical experimental data will contain some noise [12,26]. The model shapes (eigenvectors) of the original structure are continuous functions. The difference between successive displacements $x_{j-1}^{(p)}$ and $x_j^{(p)}$ ($x_{j-1,R}^{(p)}$ and $x_{jR}^{(p)}$, $x_{j-1,I}^{(p)}$ and $x_{jI}^{(p)}$) in the model with high degrees of freedom will be small. However, these small errors will lead to a large relative errors in the quantities $(d_j)_1^n=x_j-x_{j-1}$, $(d_{jR}^{(p)})_1^n=x_{jR}^{(p)}-x_{j-1,R}^{(p)}$, and $(d_{jI}^{(p)})_1^n=x_{jI}^{(p)}-x_{j-1,I}^{(p)}$ which form the equation of matrices (e.g. (31) and (32)) for Problems C. This may lead to a highly ill-posed Problem C. Then the constructed models may be sensitive to errors. Therefore, we may find an alternative method of construction to decrease the sensitivity. In this section, we will reconsider Problems C over the given noisy eigendata.

Let $w = \sum_1^n m_j$ be the total mass. Suppose that the given k eigenpairs $\{(\lambda_p, \mathbf{x}^{(p)})\}_{p=1}^k$ (as in Problem C) are corrupted by noise. When the problem is over-determined (i.e., $k \geqslant 3$), instead of solving Eqs. (31) and (32) directly, one may solve the following least-squares problems successively:

$$\min \frac{1}{2} \left\| A_{nn}\tilde{\mathbf{w}}^{(n)} - \mathbf{g}^{(n)} \right\|^2, \tag{55}$$

and

$$\min \frac{1}{2} \left\| A_{jj}\widetilde{\mathbf{w}}^{(j)} - \mathbf{g}^{(j)} \right\|^2, \quad j = n-1, n-2, \ldots, 1, \tag{56}$$

where A_{nn}, $\mathbf{g}^{(n)}$, A_{jj}, and $\tilde{\mathbf{w}}^{(j)}$ are defined in (33), (34), and (14), respectively, and

$$\mathbf{g}^{(j)} := B_{jj}\widetilde{\mathbf{w}}^{(j+1)}$$

with B_{jj} being defined in (35). By the above procedure, however, one may still find some of the updated parameters $(m_j, c_j, k_j)_1^n$ are not positive. We can observe the fact from the later numerical experiments.

On the other hand, when the problem is under-determined (i.e., $k \leqslant 2$), the systems (31) and (32) has an infinity of solutions if the conditions in Theorem 4.1 are satisfied. By using the singular value decomposition (SVD) [16], we can compute the minimum norm solutions of (31) and (32). For $j = 1, \ldots, n$, let

$$A_{jj} = \sum_{s=1}^{r} \sigma_s^{(j)} \mathbf{u}^{(j,s)} (\mathbf{v}^{(j,s)})^{\mathrm{T}}, \quad r_j = \operatorname{rank}(A_{jj})$$

is the singular value expansion, then

$$\tilde{w}_{LS}^{n} = \sum_{s=1}^{r} \frac{(\mathbf{u}^{(n,s)})^{\mathrm{T}} \mathbf{g}^{(n)}}{\sigma_s^{(n)}} \mathbf{v}^{(n,s)}$$

and

$$\tilde{w}_{LS}^{j} = \sum_{s=1}^{r} \frac{(\mathbf{u}^{(j,s)})^{\mathrm{T}} \mathbf{g}^{(j)}}{\sigma_s^{(j)}} \mathbf{v}^{(j,s)}$$

for $j = n-1, \ldots, 1$ are the minimum norm solutions. It is a form of regularization widely used in mode updating problem but rarely provides a physically feasible solution [13].

To find a stable solution, we will propose the well-known Tikhonov regularization, see for instance [11,27,28,29] for both the under-determined and over-determined problems. An natural regularization method is to add the constraint on the parameters $(m_j, c_j, k_j)_1^n$ such that the deviation between the parameters of the updated model and the estimate of the initial analytic model is minimized. Thus we replace the least squares or minimum norm solution by the minimization problems successively

$$\frac{1}{2} \left\| A_{nn}\widetilde{\mathbf{w}}^{(n)} - \mathbf{g}^{(n)} \right\|^2 + \mu_n^2 \left\| \tilde{\mathbf{w}}^{(n)} - \tilde{\mathbf{w}}_o^{(n)} \right\|^2 = \min, \tag{57}$$

and

$$\frac{1}{2}\left\|A_{jj}\tilde{\mathbf{w}}^{(j)}-\mathbf{g}^{(j)}\right\|^2+\mu_j^2\left\|\tilde{\mathbf{w}}^{(j)}-\tilde{\mathbf{w}}_o^{(j)}\right\|^2=\min, \quad j=n-1,n-2,\ldots,1, \tag{58}$$

for some regularization parameters $\{\mu_j\}_1^n$, where

$$\tilde{\mathbf{w}}_o^{(n)}:=\begin{pmatrix}\tilde{c}_n^o\\ \tilde{k}_n^o\end{pmatrix}, \quad \tilde{\mathbf{w}}_o^{(j)}:=\begin{pmatrix}\tilde{m}_j^o\\ \tilde{c}_j^o\\ \tilde{k}_j^o\end{pmatrix}, \quad j=n-1,\ldots,1.$$

Here, $(\tilde{m}_j^o:=m_j^o/m_n^o, \tilde{c}_j^o:=c_j^o/m_n^o, \tilde{k}_j^o:=k_j^o/m_n^o)_1^n$ with $(m_j^o, c_j^o, k_j^o)_1^n$ being the parameters of the initial analytic model. The solutions of (57) and (58) are given by

$$\tilde{\mathbf{w}}^{(n)}(\mu_n)=(A_{nn}^{\mathrm{T}}A_{nn}+\mu_n^2 I)^{-1}\left(A_{nn}^{\mathrm{T}}\mathbf{g}^{(n)}+\mu_n^2\tilde{\mathbf{w}}_o^{(n)}\right)$$

and

$$\tilde{\mathbf{w}}^{(j)}(\mu_j)=(A_{jj}^{\mathrm{T}}A_{jj}+\mu_j^2 I)^{-1}\left(A_{jj}^{\mathrm{T}}\mathbf{g}^{(j)}+\mu_n^2\tilde{\mathbf{w}}_o^{(j)}\right), \quad j=n-1,n-2,\ldots,1.$$

There are several criteria available for the choice of the regularization parameters $\{\mu_j\}_1^n$ for model updating [2,13]. In particular, the L-curve method [18,19,21,22] and the generalized cross validation (GCV) method [10,31] are the most widely used.

The L-curve method is the continuous curve consisting of all the points $(\|A_{nn}\tilde{\mathbf{w}}^{(n)}(\mu_n)-\mathbf{g}^{(n)}\|, \|\tilde{\mathbf{w}}^{(n)}-\tilde{\mathbf{w}}_o^{(n)}\|)$ for $\mu_n\in[0,\infty)$ (respectively, $(\|A_{nn}\tilde{\mathbf{w}}^{(j)}(\mu_j)-\mathbf{g}^{(j)}\|, \|\tilde{\mathbf{w}}^{(j)}-\tilde{\mathbf{w}}_o^{(j)}\|)$ for $\mu_j\in[0,\infty)$). In general, the curve shows the L-shape, and the optimal value of the regularization parameter μ_n (respectively, μ_j) is one that corresponds to a regularized solution near the “corner” of the L-curve because in this region there is a good compromise between achieving a small residual norm $\|A_{nn}\tilde{\mathbf{w}}^{(n)}(\mu_n)-\mathbf{g}^{(n)}\|$ (respectively, $\|A_{nn}\tilde{\mathbf{w}}^{(j)}(\mu_j)-\mathbf{g}^{(j)}\|$) and keeping the solution norm $\|\tilde{\mathbf{w}}^{(n)}-\tilde{\mathbf{w}}_o^{(n)}\|$ (respectively, $\|\tilde{\mathbf{w}}^{(j)}-\tilde{\mathbf{w}}_o^{(j)}\|$) reasonably small [18].

Suppose that the eigendata is affected by normally distributed noise, the generalized cross validation determines the optimal value of the regularization parameter μ_n (respectively, μ_j) by minimizing the functional

$$V(\mu_n)=\frac{k\|A_{nn}\tilde{\mathbf{w}}^{(n)}(\mu_n)-\mathbf{g}^{(n)}\|^2}{[\mathrm{trace}(I-A_{nn}(A_{nn}^{\mathrm{T}}A_{nn}+\mu_n^2 I)^{-1}A_{nn}^{\mathrm{T}})]^2}$$

and

$$V(\mu_j)=\frac{k\|A_j\widetilde{\mathbf{w}}^{(j)}(\mu_j)-\mathbf{g}^{(j)}\|^2}{[\mathrm{trace}(I-A_{jj}(A_{jj}^{\mathrm{T}}A_{jj}+\mu_j^2 I)^{-1}A_{jj}^{\mathrm{T}})]^2}.$$

However, in many practice, the L-curve loses its “L”-shape and the estimate of the GCV may be invariant, see [6, p. 206]. Therefore, these methods may fail to find a good value of the regularization parameters $\{\mu_j\}_1^n$.

Alternatively, we find the optimal value of the regularization parameters by solving the following constrained optimization problems successively:

$$\begin{cases}\min & \|A_{nn}\widetilde{\mathbf{w}}^{(n)}-\mathbf{g}^{(n)}\|\\ \text{subject to} & \|\tilde{\mathbf{w}}^{(n)}-\tilde{\mathbf{w}}_o^{(n)}\|\leqslant\gamma_n,\\ & \tilde{\mathbf{w}}^{(n)}\geqslant\eta\mathbf{e},\end{cases}$$

$$\begin{cases}\min & \|A_{jj}\widetilde{\mathbf{w}}^{(j)}-\mathbf{g}^{(j)}\|\\ \text{subject to} & \|\tilde{\mathbf{w}}^{(j)}-\tilde{\mathbf{w}}_o^{(j)}\|\leqslant\gamma_j, \quad j=n-1,\ldots,1,\\ & \tilde{\mathbf{w}}^{(j)}\geqslant\eta\mathbf{e},\end{cases} \tag{59}$$

or

$$\begin{cases} \min & \|\widetilde{\mathbf{w}}^{(n)} - \widetilde{\mathbf{w}}_o^{(n)}\| \\ \text{subject, to} & \|A_{nn}\widetilde{\mathbf{w}}^{(n)} - \mathbf{g}^{(n)}\| \leqslant \epsilon_n, \\ & \widetilde{\mathbf{w}}^{(n)} \geqslant \tau\mathbf{e} \end{cases}$$
$$\begin{cases} \min & \|\widetilde{\mathbf{w}}^{(j)} - \widetilde{\mathbf{w}}_o^{(j)}\| \\ \text{subject, to} & \|A_{jj}\widetilde{\mathbf{w}}^{(j)} - \mathbf{g}^{(j)}\| \leqslant \epsilon_j, \quad j = n-1, \ldots, 1, \\ & \widetilde{\mathbf{w}}^{(j)} \geqslant \tau\mathbf{e}, \end{cases} \tag{60}$$

where $\mathbf{e}$ is a vector of appropriate dimension with all the entries being ones, $\eta, \tau > 0$ are given numbers determined by the practical requirement and γ_j, ϵ_j are small positive numbers whose values depend on the noise level of the experimental data [1]. The problems (59) and (60) are constrained nonlinear minimum problem, which can be solved by the `Matlab` routine `fmincon` based on the sequential quadratic programming (SQP) method.

For the ill-posed over-determined case, to construct the positive parameters $(m_j, c_j, k_j)_1^n$, we consider the following positivity-constrained least-squares optimization problems:

$$\begin{aligned} \min \quad & \tfrac{1}{2}\left\|A_{nn}\tilde{\mathbf{w}}^{(n)} - \mathbf{g}^{(n)}\right\|^2 \\ \text{s.t.} \quad & \tilde{\mathbf{w}}^{(n)} \geqslant \delta\mathbf{e}, \end{aligned} \tag{61}$$

and

$$\begin{aligned} \min \quad & \tfrac{1}{2}\left\|A_{jj}\tilde{\mathbf{w}}^{(j)} - \mathbf{g}^{(j)}\right\|^2 \\ \text{s.t.} \quad & \widetilde{\mathbf{w}}^{(j)} \geqslant \delta\mathbf{e} \end{aligned} \tag{62}$$

for $j = n-1, n-2, \ldots, 1$. Here, $\delta > 0$ is a given number which may be determined by practical requirements.

To show how to solve above problems, we first see the general constrained least-squares problem as follows:

$$\begin{aligned} \min \quad & \tfrac{1}{2}\|H\mathbf{x} - \mathbf{h}\|^2 \\ \text{s.t.} \quad & \mathbf{x} \geqslant \alpha\mathbf{e}, \end{aligned} \tag{63}$$

where $H \in \mathbb{R}^{m\times n}$, $\mathbf{h} \in \mathbb{R}^m$, and $\alpha > 0$. Let $\mathbf{x} := \mathbf{x} - \alpha\mathbf{e}$. Then the above least-squares problem takes the following form

$$\begin{aligned} \min \quad & \tfrac{1}{2}\|H\mathbf{x} - (\mathbf{h} - \alpha H\mathbf{e})\|^2 \\ \text{s.t.} \quad & \mathbf{x} \geqslant 0. \end{aligned} \tag{64}$$

Obviously, this is a linear least squares with nonnegativity constraints. Suppose $\mathbf{x}^*$ is the unique solution of problem (64). Then the unique solution of (63) is given by $\mathbf{x}^* + \alpha\mathbf{e}$. Therefore, we can solve the positivity-constrained linear least-squares problems (61) and (62) successively by classical optimization methods for nonnegative least-squares problems (e.g. active/passive set algorithms [21,5]). Then the physical realizable parameters $(m_j, c_j, k_j)_1^n$ can be obtained by above procedure.

6. Numerical results

To show the effectiveness of our algorithm, we will give a numerical example as follows. We consider the finite element model of order $n = 10$ for free vibrating system which is governed by

Table 1
Eigenpairs of the discrete model

λ_1 -2.0214 $\mathbf{x}^{(1)}$	$\lambda_{2,3}$ $-0.0066 \pm 0.1815i$ $\mathbf{x}^{(2,3)}$	$\lambda_{4,5}$ $-0.0687 \pm 0.5642i$ $\mathbf{x}^{(4,5)}$	$\lambda_{6,7}$ $-0.1753 \pm 0.9249i$ $\mathbf{u}^{(6,7)}$	$\lambda_{8,9}$ $-0.3435 \pm 1.2335i$ $\mathbf{x}^{(8,9)}$
0.0000	0.1330 ± 0.0014	-0.3185 ∓ 0.0085	-0.0665 ∓ 0.5235	-0.1705 ∓ 0.7070
−0.0000	0.2890 ± 0.0021	-0.6140 ∓ 0.0069	-0.1231 ∓ 0.7837	-0.2375 ∓ 0.7019
0.0000	0.4902 ± 0.0021	-0.8210 ± 0.0046	-0.1060 ∓ 0.5091	-0.0257 ± 0.2651
−0.0000	0.5986 ± 0.0018	-0.8177 ± 0.0125	-0.0486 ∓ 0.1416	0.1344 ± 0.6832
0.0002	0.6968 ± 0.0019	-0.6683 ± 0.0212	0.0224 ± 0.2891	0.1903 ± 0.5438
−0.0009	0.8095 ∓ 0.0004	-0.2944 ± 0.0076	0.1346 ± 0.6454	-0.0218 ∓ 0.2262
0.0026	0.8847 ∓ 0.0020	0.0997 ∓ 0.0112	0.1098 ± 0.5512	-0.1599 ∓ 0.6350
−0.0180	0.9474 ∓ 0.0021	0.5578 ± 0.0008	0.0040 ∓ 0.0144	-0.0816 ∓ 0.3077
0.2579	0.9817 ∓ 0.0017	0.8542 ± 0.0178	-0.0768 ∓ 0.5526	0.0370 ± 0.3573
−0.4947	0.9971 ∓ 0.0029	0.9894 ∓ 0.0106	-0.2244 ∓ 0.7756	0.2858 ± 0.6172

Table 2
Relative error

RE	$s = 3$	$s = 5$	$s = 7$	$s = 9$
$\frac{\Vert\mathbf{m}_{\mathrm{appr}}-\mathbf{m}_{\mathrm{ex}}\Vert}{\Vert\mathbf{m}_{\mathrm{ex}}\Vert}$	2.9×10^{-9}	9.3×10^{-14}	1.6×10^{-13}	5.6×10^{-14}
$\frac{\Vert\mathbf{c}_{\mathrm{appr}}-\mathbf{c}_{\mathrm{ex}}\Vert}{\Vert\mathbf{c}_{\mathrm{ex}}\Vert}$	9.5×10^{-11}	2.1×10^{-14}	2.6×10^{-14}	1.8×10^{-14}
$\frac{\Vert\mathbf{k}_{\mathrm{appr}}-\mathbf{k}_{\mathrm{ex}}\Vert}{\Vert\mathbf{k}_{\mathrm{ex}}\Vert}$	1.1×10^{-10}	2.6×10^{-14}	2.3×10^{-14}	1.0×10^{-14}

Table 3
Constructed mass, damping, and stiffness parameters

j	1	2	3	4	5	6	7	8	9	10
m_j	1.3843	1.5259	1.2085	1.1464	1.4551	1.3056	1.7588	1.4446	1.3016	1.1155
c_j	4.3807	4.0153	3.1270	5.5891	5.1756	5.0087	5.9857	3.7282	3.3839	5.6784
k_j	12.7626	10.6314	7.8465	13.5589	13.3723	9.9645	11.6619	9.5670	10.0126	7.0484

Table 4
Perturbed eigenvectors

$\hat{\mathbf{x}}^{(1)}$	$\hat{\mathbf{x}}^{(2,3)}$	$\hat{\mathbf{x}}^{(4,5)}$	$\hat{\mathbf{x}}^{(6,7)}$	$\hat{\mathbf{x}}^{(8,9)}$
−0.0009	0.1340 ± 0.0019	-0.3183 ∓ 0.0078	-0.0656 ∓ 0.5239	-0.1708 ∓ 0.7070
−0.0007	0.2882 ± 0.0016	-0.6146 ∓ 0.0073	-0.1222 ∓ 0.7839	-0.2382 ∓ 0.7027
−0.0003	0.4897 ± 0.0016	-0.8213 ± 0.0047	-0.1069 ∓ 0.5084	-0.0251 ± 0.2645
−0.0006	0.5979 ± 0.0016	-0.8173 ± 0.0116	-0.0493 ∓ 0.1415	0.1352 ± 0.6833
0.0003	0.6962 ± 0.0019	-0.6687 ± 0.0219	0.0234 ± 0.2901	0.1900 ± 0.5444
−0.0009	0.8085 ± 0.0006	-0.2934 ± 0.0082	0.1341 ± 0.6451	-0.0215 ∓ 0.2269
0.0035	0.8838 ∓ 0.0028	0.0995 ∓ 0.0102	0.1091 ± 0.5505	-0.1607 ∓ 0.6345
−0.0171	0.9481 ∓ 0.0019	0.5573 ± 0.0000	0.0046 ∓ 0.0149	-0.0812 ∓ 0.3082
0.2578	0.9808 ∓ 0.0012	0.8543 ± 0.0181	-0.0766 ∓ 0.5525	0.0364 ± 0.3578
−0.4948	0.9972 ∓ 0.0030	0.9895 ∓ 0.0097	-0.2238 ∓ 0.7746	0.2853 ± 0.6167

Table 5
Constructed mass and damping parameters

m_j	Exact	$s = 3$	$s = 5$	$s = 7$	$s = 9$	c_j	Exact	$s = 3$	$s = 5$	$s = 7$	$s = 9$
1	1.4360	2.4850	0.8194	1.5355	1.4389	1	4.3780	4.4897	4.6555	4.3227	4.3538
2	1.5401	−2.3071	1.7277	1.4646	1.5282	2	4.0110	5.0003	4.3081	3.9459	3.9552
3	1.1141	3.2693	1.1624	1.1760	1.1249	3	3.1299	3.6256	3.4385	3.0901	3.1263
4	1.0754	−3.0153	1.1483	1.0041	1.0629	4	5.6259	6.8113	5.9581	5.6565	5.6443
5	1.4964	3.7243	1.5204	1.5229	1.5057	5	5.2197	6.8906	5.3820	5.2141	5.2535
6	1.3537	5.2184	1.5387	1.3474	1.3504	6	5.0297	4.6982	5.3242	4.9680	5.0151
7	1.8337	3.1366	1.6805	1.8116	1.8309	7	5.9495	3.9867	6.3193	5.8908	5.9908
8	1.3974	1.7486	1.5538	1.3985	1.4018	8	3.6815	0.2915	3.9451	3.6603	3.6896
9	1.2314	−0.2668	1.2761	1.2259	1.2332	9	3.4181	−0.5826	3.5109	3.4021	3.4102
10	1.1680	−0.3468	1.2190	1.1597	1.1693	10	5.9454	−1.6932	6.2091	5.9120	5.9582

Table 6
Constructed stiffness parameters

k_j	Exact	$s = 3$	$s = 5$	$s = 7$	$s = 9$
1	12.7586	13.6628	13.2621	12.6760	12.7209
2	10.6233	11.9998	11.2915	10.5859	10.6227
3	7.8552	8.8830	8.2406	7.7914	7.8335
4	13.6456	16.7363	14.3518	13.6887	13.6628
5	13.4818	17.7558	14.1613	13.3454	13.4253
6	10.0050	9.9822	10.5008	9.9262	10.0035
7	11.5915	6.1531	12.2609	11.5661	11.6238
8	9.4480	0.6472	9.9126	9.3929	9.4562
9	10.1156	−2.8647	10.5204	10.0472	10.1181
10	7.3799	−2.0449	7.7082	7.3428	7.3947

Table 7
Relative approximation error

RE	$s = 3$	$s = 5$	$s = 7$	$s = 9$
$\frac{\\|\mathbf{m}_{\text{appr}}-\mathbf{m}_{\text{ex}}\\|}{\\|\mathbf{m}_{\text{ex}}\\|}$	1.8240	0.1635	0.0368	0.0054
$\frac{\\|\mathbf{c}_{\text{appr}}-\mathbf{c}_{\text{ex}}\\|}{\\|\mathbf{c}_{\text{ex}}\\|}$	0.6503	0.0584	0.0092	0.0057
$\frac{\\|\mathbf{k}_{\text{appr}}-\mathbf{k}_{\text{ex}}\\|}{\\|\mathbf{k}_{\text{ex}}\\|}$	0.5778	0.0502	0.0064	0.0024

Eq. (1), where the mass, damping, and stiffness matrices are given by (3)–(5) with $(m_j, c_j, k_j)_1^{10}$ generated randomly

$$\begin{cases} (m_j)_1^{10} = (1.4360, 1.5401, 1.1141, 1.0754, 1.4964, 1.3537, 1.8337, 1.3974, 1.2314, 1.1680), \\ (c_j)_1^{10} = (4.3780, 4.0110, 3.1299, 5.6259, 5.2197, 5.0297, 5.9495, 3.6815, 3.4181, 5.9454), \\ (k_j)_1^{10} = (12.7586, 10.6233, 7.8552, 13.6456, 13.4818, 10.0050, 11.5915, 9.4480, 10.1156, 7.3799). \end{cases}$$

The nine eigenvalues λ_j with lowest absolute values of imaginary parts and their corresponding eigenvectors $\mathbf{x}^{(j)}$ are listed in Table 1. Obviously, the total mass is $w = \sum_1^{10} m_j = 13.6462$. Also, suppose the estimate of the parameters $(m_j^o, c_j^o, k_j^o)_1^{10}$ for the initial analytic model $P_o(\lambda) = \lambda^2 M^o + \lambda C^o + K^o$ are obtained by perturbing the parameters $(m_j, c_j, k_j)_1^{10}$ by a uniform distribution between -0.1 and 0.1, i.e.,

Table 8
Constructed mass, damping, and stiffness parameters

m_j	Exact	(1)	(2)	c_j	Exact	(1)	(2)	k_j	Exact	(1)	(2)
1	1.4360	−0.1026	0.6802	1	4.3780	0.0254	−21.656	1	12.7586	−0.0126	−1.6275
2	1.5401	0.0821	1.5439	2	4.0110	0.0051	−19.035	2	10.6233	−0.0025	−1.5831
3	1.1141	−0.1202	1.6454	3	3.1299	−0.0193	−14.763	3	7.8552	0.0095	−1.6824
4	1.0754	0.1305	11.117	4	5.6259	−0.0151	−30.004	4	13.6456	0.0075	−9.1197
5	1.4964	−0.0209	39.880	5	5.2197	0.3106	−45.392	5	13.4818	−0.1536	−41.662
6	1.3537	2.0492	125.76	6	5.0297	−0.8567	−93.111	6	10.0050	0.4238	−140.90
7	1.8337	4.3973	−770.43	7	5.9495	−1.5522	282.49	7	11.5915	0.7679	504.96
8	1.3974	5.3708	2129.18	8	3.6815	−1.7861	−452.06	8	9.4480	0.8836	−1354.77
9	1.2314	1.0531	−3342.35	9	3.4181	−0.2872	250.86	9	10.1156	0.1421	1066.78
10	1.1680	0.8069	1816.66	10	5.9454	1.2741	1086.02	10	7.3799	−0.6303	10693.26

Table 9
Constructed mass, damping, and stiffness parameters by (59)

	ξ										
	10.0	1.0	0.1		10.0	1.0	0.1		10.0	1.0	0.1
(a)											
m_1	0.6716	0.6718	0.6725	c_1	5.9515	6.4908	7.1205	k_1	0.6716	0.6718	16.4367
m_2	0.6716	0.6718	0.6725	c_2	0.6716	5.3699	5.3753	k_2	16.7449	14.3815	14.4385
m_3	0.6716	0.6718	0.6725	c_3	4.0936	4.0946	3.6599	k_3	10.7538	10.7564	10.9845
m_4	0.6716	0.6718	0.6725	c_4	22.9568	22.9468	9.2953	k_4	10.7094	10.7171	17.4947
m_5	1.8843	1.8820	1.8841	c_5	8.0163	8.0193	8.0260	k_5	17.6768	17.6805	17.6992
m_6	2.8159	2.8160	2.8074	c_6	6.4305	6.4307	6.4420	k_6	13.7418	13.7450	13.7566
m_7	1.5366	1.5369	1.5360	c_7	8.3387	8.3407	8.3435	k_7	15.4397	15.4434	15.4620
m_8	1.9582	1.9587	1.9607	c_8	5.0697	5.0710	5.0761	k_8	12.5739	12.5769	12.5897
m_9	1.4213	1.4217	1.4231	c_9	4.8498	4.8510	4.8559	k_9	13.4649	13.4682	13.4819
m_{10}	1.3433	1.3436	1.3450	c_{10}	7.6350	7.6369	7.6447	k_{10}	10.0967	10.0991	10.1094
(b)											
m_1	1.3740	1.3748	1.3769	c_1	4.1034	4.1384	4.0506	k_1	12.6506	12.6482	12.7014
m_2	1.4685	1.4692	1.4720	c_2	4.4101	4.4402	4.3678	k_2	10.7353	10.7330	10.7790
m_3	1.2075	1.2078	1.2101	c_3	3.1415	3.1645	3.1090	k_3	7.8358	7.8339	7.8684
m_4	1.1379	1.1367	1.1400	c_4	5.4128	5.4559	5.3515	k_4	13.5844	13.5813	13.6430
m_5	1.5483	1.5430	1.5488	c_5	5.0303	5.0793	4.9610	k_5	13.3201	13.3208	13.3831
m_6	1.7534	1.7544	1.7551	c_6	4.4251	4.4282	4.3688	k_6	9.6126	9.6191	9.6665
m_7	1.1942	1.1950	1.1934	c_7	7.1395	7.1444	7.0693	k_7	11.2248	11.2324	11.3046
m_8	1.5435	1.5446	1.5870	c_8	2.9270	2.9290	2.7962	k_8	9.4514	9.4579	9.4843
m_9	1.4276	1.4285	1.3695	c_9	2.7487	2.7505	2.5008	k_9	10.1546	10.1615	10.2656
m_{10}	0.9913	0.9920	0.9934	c_{10}	5.9221	5.9262	5.9285	k_{10}	5.8354	5.8394	6.3878
	ξ										
	$s=5$	$s=7$	$s=9$		$s=5$	$s=7$	$s=9$		$s=5$	$s=7$	$s=9$
(c)											
m_1	0.4941	1.5000	1.4389	c_1	4.6826	4.3532	4.4082	k_1	13.2828	12.8228	12.8827
m_2	1.9963	1.5127	1.5596	c_2	4.2821	3.9564	4.0023	k_2	11.1954	10.7051	10.7520
m_3	0.9373	1.1583	1.1264	c_3	3.4158	3.2033	3.1717	k_3	8.1754	7.8851	7.9491
m_4	1.3352	1.0462	1.0700	c_4	6.3742	5.7633	5.6860	k_4	14.2226	13.7912	13.7642
m_5	1.3011	1.5059	1.5175	c_5	5.6609	5.3288	5.2914	k_5	14.0379	13.4978	13.5238
m_6	1.7742	1.3855	1.3594	c_6	5.3897	5.0714	5.0521	k_6	10.3688	10.0029	10.0782
m_7	1.3109	1.7942	1.8300	c_7	6.4182	6.0542	5.9664	k_7	12.2295	11.5413	11.6116
m_8	2.4160	1.4189	1.3998	c_8	3.8695	3.7462	3.7377	k_8	9.3957	9.3674	9.4402
m_9	1.0930	1.2202	1.2350	c_9	3.2426	3.5134	3.4487	k_9	8.9550	10.0292	10.1005
m_{10}	0.9882	1.1042	1.1096	c_{10}	5.2249	5.8422	5.8750	k_{10}	6.6321	7.4072	7.4394

$$
\begin{cases}
(m_j^o)_1^{10} = (1.3855, 1.5273, 1.2096, 1.1474, 1.4565, 1.3068, 1.7604, 1.4459, 1.3028, 1.1166),\\
(c_j^o)_1^{10} = (4.3833, 4.0610, 3.1629, 5.7187, 5.1413, 5.1159, 5.9852, 3.7781, 3.4411, 5.9615),\\
(k_j^o)_1^{10} = (12.6718, 10.7033, 7.9485, 13.5974, 13.5682, 10.0667, 11.6040, 9.3586, 10.1077, 7.3791).
\end{cases}
$$

We first reconstruct the mass, damping, and stiffness parameters from the over-determined exact eigendata $\{(\lambda_j, \mathbf{x}^{(j)})\}_1^s$ for $s = 3, 5, 7, 9$. Obviously, these data satisfy the solvability conditions in Corollary 4.2. Table 2 shows the relative error between the constructed and the exact parameters, where $\mathbf{m}$, $\mathbf{c}$, and $\mathbf{k}$ are the required parameter vectors defined by

$$
\mathbf{m} = (m_1, m_2, \dots, m_n)^{\mathrm{T}}, \quad \mathbf{c} = (c_1, c_2, \dots, c_n)^{\mathrm{T}}, \quad \text{and} \quad \mathbf{k} = (k_1, k_2, \dots, k_n)^{\mathrm{T}}
$$

Table 10
Constructed mass, damping, and stiffness parameters by (60)

	κ										
	10^{-3}	10^{-6}	10^{-9}		10^{-3}	10^{-6}	10^{-9}		10^{-3}	10^{-6}	10^{-9}
(a)											
m_1	0.7810	0.7879	0.5819	c_1	29.7823	30.1123	18.0353	k_1	8.4478	8.4877	6.6086
m_2	0.7810	0.7879	0.5819	c_2	0.7810	0.7879	0.5819	k_2	124.6923	126.1328	63.6041
m_3	0.7810	0.7879	0.5819	c_3	0.7810	0.7879	0.5819	k_3	59.1674	59.8120	22.8038
m_4	0.7810	0.7879	0.5819	c_4	26.8368	27.0965	10.0238	k_4	12.3826	12.4791	12.4179
m_5	2.1921	2.2115	0.5819	c_5	9.2612	9.3428	8.0198	k_5	20.5859	20.7660	13.0430
m_6	2.8822	2.8911	7.2201	c_6	7.6400	7.7138	2.0825	k_6	15.8988	16.0348	12.0611
m_7	0.9806	0.9453	0.5819	c_7	9.9737	10.0768	10.3252	k_7	17.8171	17.9659	10.1025
m_8	1.2931	1.2541	0.5819	c_8	6.2226	6.2939	3.4103	k_8	14.4603	14.5785	9.6396
m_9	1.6120	1.6170	1.1892	c_9	5.4905	5.5411	4.0736	k_9	15.7319	15.8685	11.7187
m_{10}	1.5621	1.5758	1.1638	c_{10}	8.8839	8.9564	10.9084	k_{10}	11.7385	11.8441	17.4266
(b)											
m_1	1.3791	1.3741	1.3734	c_1	4.1614	4.1040	4.1040	k_1	12.6642	12.6497	12.6503
m_2	1.4828	1.4680	1.4677	c_2	4.3408	4.4104	4.4106	k_2	10.7458	10.7346	10.7352
m_3	1.2132	1.2080	1.2079	c_3	3.1754	3.1418	3.1419	k_3	7.8428	7.8352	7.8358
m_4	1.1166	1.1375	1.1388	c_4	5.6254	5.4135	5.4134	k_4	13.5962	13.5833	13.5837
m_5	1.5395	1.5505	1.5510	c_5	5.1170	5.0304	5.0300	k_5	13.3558	13.3177	13.3171
m_6	1.7342	1.7542	1.7533	c_6	4.5245	4.4242	4.4240	k_6	9.6396	9.6087	9.6085
m_7	1.2192	1.1935	1.1945	c_7	7.1456	7.1369	7.1365	k_7	11.2530	11.2189	11.2183
m_8	1.5417	1.5438	1.5409	c_8	3.0396	2.9265	2.9270	k_8	9.4551	9.4453	9.4471
m_9	1.4253	1.4252	1.4272	c_9	2.8159	2.7495	2.7511	k_9	10.1737	10.1485	10.1545
m_{10}	0.9945	0.9914	0.9915	c_{10}	5.9440	5.9272	5.9274	k_{10}	5.8673	5.8360	5.8363
	κ										
	$s=5$	$s=7$	$s=9$		$s=5$	$s=7$	$s=9$		$s=5$	$s=7$	$s=9$
(c)											
m_1	1.3962	1.3962	1.3962	c_1	4.4170	4.4170	4.4170	k_1	12.7690	12.7690	12.7690
m_2	1.5390	1.5390	1.5390	c_2	4.0922	4.0922	4.0922	k_2	10.7854	10.7854	10.7854
m_3	1.2189	1.2189	1.2189	c_3	3.1872	3.1872	3.1872	k_3	8.0095	8.0095	8.0095
m_4	1.1562	1.1562	1.1562	c_4	5.7626	5.7626	5.7626	k_4	13.7017	13.7017	13.7017
m_5	1.4677	1.4677	1.4677	c_5	5.1808	5.1808	5.1808	k_5	13.6723	13.6723	13.6723
m_6	1.3169	1.3169	1.3169	c_6	5.1552	5.1552	5.1552	k_6	10.1440	10.1440	10.1440
m_7	1.7739	1.7739	1.7739	c_7	6.0311	6.0311	6.0311	k_7	11.6931	11.6931	11.6931
m_8	1.4570	1.4570	1.4570	c_8	3.8071	3.8071	3.8071	k_8	9.4304	9.4304	9.4304
m_9	1.3128	1.3128	1.3128	c_9	3.4675	3.4675	3.4675	k_9	10.1852	10.1852	10.1852
m_{10}	1.0077	1.0077	1.0077	c_{10}	6.0072	6.0072	6.0072	k_{10}	7.4357	7.4357	7.4357

and `RE.`, `ex` and `appr` denote the relative error, the exact and approximated solution, respectively. We can see that the expected parameters are constructively obtained given the exact data.

Next, we will reconstruct the mass, damping, and stiffness parameters from two complex conjugate eigenpairs by Algorithm 4.6. Without loss of generality, we use $\{(\lambda_j, \mathbf{x}^{(j)})\}_2^3$ as the given data. The constructed mass, damping, and stiffness parameters are listed in Table 3. Table 3 shows the constructed parameters is different from the true solution but physically feasible.

Table 11
Constructed mass and damping parameters

m_j	Exact	$s=3$	$s=5$	$s=7$	$s=9$	c_j	Exact	$s=3$	$s=5$	$s=7$	$s=9$
1	1.4360	0.0177	0.8194	1.5355	1.4389	1	4.3780	4.9316	4.6555	4.3227	4.3538
2	1.5401	0.0177	1.7277	1.4646	1.5282	2	4.0110	5.3184	4.3081	3.9459	3.9552
3	1.1141	0.2082	1.1624	1.1760	1.1249	3	3.1299	3.9792	3.4385	3.0901	3.1263
4	1.0754	0.0177	1.1483	1.0041	1.0629	4	5.6259	7.3596	5.9581	5.6565	5.6443
5	1.4964	9.4682	1.5204	1.5229	1.5057	5	5.2197	5.2819	5.3820	5.2141	5.2535
6	1.3537	2.6541	1.5387	1.3474	1.3504	6	5.0297	2.0657	5.3242	4.9680	5.0151
7	1.8337	1.1826	1.6805	1.8116	1.8309	7	5.9495	1.4690	6.3193	5.8908	5.9908
8	1.3974	0.0177	1.5538	1.3985	1.4018	8	3.6815	0.6473	3.9451	3.6603	3.6896
9	1.2314	0.0272	1.2761	1.2259	1.2332	9	3.4181	0.0593	3.5109	3.4021	3.4102
10	1.1680	0.0353	1.2190	1.1597	1.1693	10	5.9454	0.1725	6.2091	5.9120	5.9582

Table 12
Constructed stiffness parameters

k_j	Exact	$s=3$	$s=5$	$s=7$	$s=9$
1	12.7586	13.7307	13.2621	12.6760	12.7209
2	10.6233	11.9227	11.2915	10.5859	10.6227
3	7.8552	9.0880	8.2406	7.7914	7.8335
4	13.6456	16.8426	14.3518	13.6887	13.6628
5	13.4818	12.5113	14.1613	13.3454	13.4253
6	10.0050	3.6957	10.5008	9.9262	10.0035
7	11.5915	1.5226	12.2609	11.5661	11.6238
8	9.4480	0.2074	9.9126	9.3929	9.4562
9	10.1156	0.2918	10.5204	10.0472	10.1181
10	7.3799	0.2083	7.7082	7.3428	7.3947

Now, we consider the problem with the noisy data. We perturb the eigenvectors $\mathbf{x}^{(j)}$ with random numbers uniformly distributed on the interval $(-0.001, 0.001)$. The perturbed eigenvectors $\hat{\mathbf{x}}^{(j)}$ are shown in Table 4.

For the purpose of comparison, we reconstruct the mass, damping, and stiffness parameters from the over-determined noisy data $\{(\lambda_j, \widehat{\mathbf{x}}^{(j)})\}_1^s$ with $s = 3, 5, 7, 9$. The constructed least-squares solutions are listed in Tables 5 and 6. We can easily observe the improvement of the estimation from over-determined data. Except the case when $s = 3$, the constructed mass, damping, stiffness are all physical realistic for other values of s. Finally, the relative error are shown in Table 7. Table 7 displays that the relative error becomes expectedly smaller with the increase of the number of over-determined eigenpairs. On the other hand, if we construct these parameters from the under-determined noisy data: 1) $(\lambda_1, \hat{\mathbf{x}}^{(1)})$ or 2) $\{(\lambda_j, \hat{\mathbf{x}}^{(j)})\}_2^3$. Then the minimum norm solutions are displayed in Table 8. It is seen from Table 8 that both the minimum norm solutions are not physically feasible.

Now, we find a stable solution for the ill-posed problem. In our numerical experiments, we observe that the L-curve method lost its "L"-shape and the GCV method retained the invariance property. These drawbacks make it difficult to choose good values of the regularization parameters $\{\mu_j\}_1^n$ for our problems. So we choose the regularization parameters $\{\mu_j\}_1^n$ by solving (59) or (60). We consider the following under-determined and over-determined cases, for example, from the noisy data: (a) $(\lambda_1, \widehat{\mathbf{x}}^{(1)})$, (b) $\{(\lambda_j, \hat{\mathbf{x}}^{(j)})\}_2^3$, and (c) $\{(\lambda_j, \hat{\mathbf{x}}^{(j)})\}_1^s$ for $s = 5, 7, 9$. For simplicity, we

Table 13
Perturbed eigenvectors

$\tilde{\mathbf{x}}^{(1)}$	$\tilde{\mathbf{x}}^{(2,3)}$	$\tilde{\mathbf{x}}^{(4,5)}$	$\tilde{\mathbf{x}}^{(6,7)}$	$\tilde{\mathbf{x}}^{(8,9)}$
−0.0937	0.2295 ± 0.0505	-0.2947 ± 0.0565	0.0280 ∓ 0.5687	-0.2003 ∓ 0.7026
−0.0657	0.2149 ∓ 0.0404	-0.6776 ∓ 0.0430	-0.0247 ∓ 0.8052	-0.3091 ∓ 0.7787
−0.0272	0.4396 ∓ 0.0455	-0.8579 ± 0.0181	-0.1953 ∓ 0.4313	0.0295 ± 0.2101
−0.0600	0.5310 ∓ 0.0159	-0.7770 ∓ 0.0736	-0.1188 ∓ 0.1311	0.2195 ± 0.6897
0.0139	0.6319 ± 0.0040	-0.7165 ± 0.0898	0.1207 ± 0.3885	0.1624 ± 0.6017
−0.0023	0.7129 ± 0.0909	-0.1993 ± 0.0686	0.0803 ± 0.6175	0.0023 ∓ 0.2973
0.0913	0.7918 ∓ 0.0782	0.0806 ± 0.0827	0.0459 ± 0.4746	-0.2459 ∓ 0.5839
0.0762	1.0252 ± 0.0167	0.5053 ∓ 0.0810	0.0685 ∓ 0.0638	-0.0432 ∓ 0.3531
0.2470	0.8899 ± 0.0563	0.8676 ± 0.0487	-0.0597 ∓ 0.5425	-0.0261 ± 0.4079
−0.5053	1.0042 ∓ 0.0142	1.0013 ± 0.0796	-0.1633 ∓ 0.6813	0.2378 ± 0.5662

Table 14
Constructed mass parameters

m_j	Exact	LS				LSP			
		$s = 3$	$s = 5$	$s = 7$	$s = 9$	$s = 3$	$s = 5$	$s = 7$	$s = 9$
1	1.4360	12.9057	10.9010	6.8952	1.7398	0.7274	1.0949	1.0949	1.0949
2	1.5401	−8.2321	−10.0628	−4.8752	−0.9282	0.7274	1.0949	1.0949	1.0949
3	1.1141	8.7035	13.3456	5.9059	2.2882	0.7274	1.0949	1.0949	1.0949
4	1.0754	−7.8355	−12.6486	−4.9820	−0.9015	0.7274	1.0949	1.0949	1.0949
5	1.4964	7.4295	14.0211	6.0671	2.5244	0.7274	1.5467	1.5467	1.5467
6	1.3537	0.6405	−12.0545	−2.4680	0.4464	0.7274	1.0949	1.0949	1.0949
7	1.8337	0.0262	6.2632	4.4962	2.0392	5.0765	1.0949	1.0949	1.0949
8	1.3974	−0.0016	−2.1849	−2.2597	1.8613	0.7274	1.0949	1.0949	1.0949
9	1.2314	0.0047	2.8448	2.5797	2.3767	2.0229	2.2457	2.2457	2.2457
10	1.1680	0.0052	3.2213	2.2868	2.1998	1.4549	2.1897	2.1897	2.1897

set $\eta = \tau = 0.5 > 0$. Table 9 includes the constructed mass, damping, and stiffness parameters for $\gamma_j = \xi \times \|\tilde{\mathbf{w}}_o^{(j)}\|$ with the varying value of ξ for Case (a) and (b) and the fixed value $\xi = 0.1$ for Case (c). Table 10 includes the constructed mass, damping, and stiffness parameters for $\epsilon_j = \kappa \times (\|A_{jj}\tilde{\mathbf{w}}_o^{(j)}\| + \|\mathbf{g}^{(j)}\|)$ with the varying value of κ for Case (a) and (b) and the fixed value $\kappa = 0.1$ for Case (c). We can see from Tables 9 and 10 that an expected stable and physically feasible solution is obtained in each case. We also observe from our numerical experiments the fact that the regularization methods based on (59) or (60) work effectively even for the given noisy eigendata with much larger errors.

Next, we reconstruct the parameters $(m_j, c_j, k_j)_1^{10}$ by solving above constrained least-squares optimization problems (61) and (62) with the over-determined data $\{(\lambda_j, \hat{\mathbf{x}}^{(j)})\}_1^s$ for $s = 3, 5, 7, 9$. For simplicity, we set $\delta = 0.5 > 0$. Tables 11 and 12 include the constructed mass, damping, and stiffness parameters, respectively. From Tables 11 and 12, we can see that all the constructed parameters are positive for all values of s. This shows that the algorithm proposed in Section 5 yield acceptable results.

To further illustrate the effectiveness of the constrained least-squares optimization method, we change the eigenvectors $\{\mathbf{x}^{(j)}\}_1^9$ with large errors. For example, we perturb these eigenvectors by a uniform distribution between -0.1 and 0.1. The perturbed eigenvectors $\tilde{\mathbf{x}}^{(j)}$ are shown in Table 13. Then we reconstruct the model parameters $(m_j, c_j, k_j)_1^{10}$ by solving both the

Table 15
Constructed damping parameters

c_j	Exact	LS				LSP			
		$s=3$	$s=5$	$s=7$	$s=9$	$s=3$	$s=5$	$s=7$	$s=9$
1	4.3780	−1.8481	−6.1334	−0.2534	−0.0546	6.7868	9.7403	9.7403	9.7403
2	4.0110	33.9379	−5.0655	0.5326	−1.0893	0.7274	1.0949	1.0949	1.0949
3	3.1299	1.3655	−15.1989	−1.7273	1.2321	0.7274	1.0949	1.0949	1.0949
4	5.6259	−6.6282	27.1637	6.1650	5.2688	14.3470	6.4661	6.4661	6.4661
5	5.2197	−3.2575	−16.1711	2.4353	5.0864	10.2324	2.9924	2.9924	2.9924
6	5.0297	−1.4070	9.0094	3.1657	4.5888	0.7274	6.5217	6.5217	6.5217
7	5.9495	0.0985	15.2712	7.4984	12.8800	24.3331	13.1154	13.1154	13.1154
8	3.6815	−0.0104	13.6657	5.1899	7.1767	0.7274	11.6797	11.6797	11.6797
9	3.4181	−0.0210	−2.1159	6.2126	4.0477	0.7274	1.0949	1.0949	1.0949
10	5.9454	0.0123	12.2913	11.3894	11.0405	3.4087	8.3552	8.3552	8.3552

Table 16
Constructed stiffness parameters

k_j	Exact	LS				LSP			
		$s=3$	$s=5$	$s=7$	$s=9$	$s=3$	$s=5$	$s=7$	$s=9$
1	12.7586	2.4771	2.3744	−0.2132	1.7622	0.7274	4.5823	4.5823	4.5823
2	10.6233	−1.0043	7.5856	4.0691	4.6821	1.6018	5.0356	5.0356	5.0356
3	7.8552	2.2522	−1.9176	−0.1602	1.9632	7.7955	3.5757	3.5757	3.5757
4	13.6456	5.6359	6.1022	13.9451	10.3818	18.1074	1.0949	1.0949	1.0949
5	13.4818	5.1054	9.2500	2.5029	6.0708	16.3208	7.9030	7.9030	7.9030
6	10.0050	0.3162	11.1928	6.4722	9.8678	15.6725	3.7800	3.7800	3.7800
7	11.5915	0.0074	9.3144	11.9589	12.5322	1.0196	9.9123	9.9123	9.9123
8	9.4480	0.0065	7.9727	12.4826	12.2428	1.6321	7.1326	7.1326	7.1326
9	10.1156	−0.0095	13.4619	15.2008	15.0072	0.7274	9.8310	9.8310	9.8310
10	7.3799	0.0031	11.5439	13.4617	12.5696	0.8555	7.8472	7.8472	7.8472

unconstrained and constrained least-squares optimization problems with the over-determined data $\{(\lambda_j, \tilde{\mathbf{x}}^{(j)})\}_1^s$ for $s = 3, 5, 7, 9$. For simplicity, we still set $\delta = 0.5 > 0$. Tables 14–16 contain the constructed mass, damping, and stiffness parameters, respectively. Here, LS. and LSP. denote the least squares method and the optimization method for positivity-constrained least squares optimization problems proposed in Section 5, respectively. We observe from Tables 14–16 that the positivity-constrained least squares solutions are positive, but the unconstrained least squares solutions are no longer physical realizable.

7. Conclusions

In this paper, we have studied the reconstruction of a free vibration system where the mass, damping, and stiffness matrices are all symmetric tridiagonal. This system can be constructed from two real eigenvalues and three real eigenvectors or a real eigenvector and two complex conjugate eigenpairs. For large dimensional model, the construction based on these data is sensitive to the perturbations. To reduce the sensitivity, we construct a least-squares solution based on the over-determined eigendata. We also discuss the solvability for the under-determined and over-determine problems. In particular, we establish the solvability conditions on the given two real or complex conjugate eigenpairs for the existence of a physically feasible solution of the under-determined problem. However, these methods do not theoretically ensure that the required mass, damping, and stiffness parameters are positive. Finally, we discussed the physical realizability of the required model. Given the under-determined noisy eigendata, we propose the well-known Tikhonov regularization method, while for the over-determined noised eigendata, we take a set of linear least-squares optimization problems with positivity-constraint.

Finally, we should point out that the problem of finding a necessary and sufficient condition on the exact and over-determined eigendata so that the constructed solution is physically realizable is an interesting topic which remains to be investigated.

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